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Flat $(2,3,5)$-Distributions and Chazy's Equations

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Abstract. In the geometry of generic 2-plane fields on 5-manifolds, the local equivalence problem was solved by Cartan who also constructed the fundamental curvature invariant. For generic 2-plane fields or $(2,3,5)$-distributions determined by a single function of the form $F(q)$, the vanishing condition for the curvature invariant is given by a 6^{th} order nonlinear ODE. Furthermore, An and Nurowski showed that this ODE is the Legendre transform of the 7^{th} order nonlinear ODE described in Dunajski and Sokolov. We show that the 6^{th} order ODE can be reduced to a 3^{rd} order nonlinear ODE that is a generalised Chazy equation. The 7^{th} order ODE can similarly be reduced to another generalised Chazy equation, which has its Chazy parameter given by the reciprocal of the former. As a consequence of solving the related generalised Chazy equations, we obtain additional examples of flat $(2,3,5)$-distributions not of the form $F(q)=q^m$. We also give 4-dimensional split signature metrics where their twistor distributions via the An–Nurowski construction have split G_2 as their group of symmetries.

Key words: generic rank two distribution in dimension five; conformal geometry; Chazy's equations

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1 Introduction

The following 6^{th} order nonlinear ODE

$$10F^{(6)}F''^3-80F''^2F^{(3)}F^{(5)}-51F''^2{F^{(4)}}^2+336F''{F^{(3)}}^2F^{(4)}-224{F^{(3)}}^4=0, \tag{1.1}$$

arises in [6, Corollary 2.1] in the study of generic 2-plane fields on 5-manifolds. The genericity condition here means $F''(q)\neq 0$ in (1.1). This ODE arises as the integrability condition for generic 2-plane fields on 5 manifolds determined by a function of a single variable of the form $F(q)$. A generic 2-plane field $\mathcal{D}$ on a 5-manifold M is a maximally non-integrable rank 2 distribution. For further details, see [6, 18, 24, 25]. This determines a filtration of the tangent bundle given by

$$\mathcal{D}\subset[\mathcal{D},\mathcal{D}]\subset[[\mathcal{D},\mathcal{D}],\mathcal{D}]=TM.$$

The distribution $[\mathcal{D},\mathcal{D}]$ has rank 3 while the full tangent space TM has rank 5, hence such a geometry is also known as a $(2,3,5)$-distribution. Let M_{xyzpq} denote the 5-dimensional manifold with local coordinates given by (x,y,z,p,q). The generic 2-plane field or rank 2 distribution determined by a function $F(q)$ of a single variable with $F''(q)\neq 0$ is given by

$$\mathcal{D}=\operatorname{span}\{\partial_q,\partial_x+p\partial_y+q\partial_p+F(q)\partial_z\}.$$

The fundamental Cartan curvature invariant of this distribution is computed in [6] and is found to be the term in the left hand side of (1.1). It is known that equation (1.1) vanishes when

$F(q) = q^m$ and $m \in \left\{-1, \frac{1}{3}, \frac{2}{3}, 2\right\}$. In these cases, the vanishing of the fundamental curvature invariant associated to the distribution $\mathcal{D}$ on M_{xyzpq} implies that the group of local symmetries is the maximal possible given by the split real form of G_2. This is the result of [6, Corollary 2.1]. The authors of [6] call such generic 2-plane fields with vanishing Cartan curvature invariant symmetric and it is known that such symmetric or flat distributions are locally equivalent to the flat model $F(q) = q^2$. Nonetheless we are interested in the general solution to (1.1) and it turns out that the ODE can be solved completely and is related to the generalised Chazy equation. To see this, let $E(q) = F''(q)$ so that the ODE becomes 4th order:

$$10E^{(4)}E^3 - 80E^2E'E''' - 51E^2E''^2 + 336EE'^2E'' - 224E'^4 = 0.$$

Working locally on an open set of M_{xyzpq}, we may assume that $E(q)$ is positive on that open set. Making the substitution $E(q) = \mathrm{e}^{G(q)}$ (if $E(q) < 0$, take $E(q) = -\mathrm{e}^{G(q)}$ instead) gives

$$\mathrm{e}^{4G(q)}\left(10G'''' - 40G'''G' - 21(G'')^2 + 54G''(G')^2 - 9(G')^4\right) = 0$$

and taking $G'(q) = j(q)$ gives a 3rd order ODE

$$10j''' - 40j''j - 21(j')^2 + 54j'j^2 - 9j^4 = 0.$$

Rescaling the ODE by taking $j(q) = \frac{I(q)}{2}$, we can put it into the normal form for the generalised Chazy equation (see [14])

$$I''' - 2I''I + 3(I')^2 - \frac{4}{36 - \left(\frac{2}{3}\right)^2}\left(6I' - I^2\right)^2 = 0 \tag{1.2}$$

with the Chazy parameter $k^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$. The generalised Chazy equation can be solved completely and the solutions give us new families of flat $(2,3,5)$-distributions that are not of the form $F(q) = q^m$. In this article we first review the solutions to Chazy's equations in Sections 2 and 3. In Section 4 we discuss the relationship between (1.1) and a 7th order ODE studied by Dunajski and Sokolov in [16] and also exhibit a Legendre transform that relates equation (1.2) to another generalised Chazy equation with the Chazy parameter given by $k^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$. We compute the solutions to (1.2) in Section 5 and present examples of flat $(2,3,5)$-distributions in Section 6 using Nurowski's metric. These examples are all explicit. In [5], the authors associated to split signature conformal structures on a 4-manifold a circle bundle with the natural structure of a $(2,3,5)$-distribution. This construction encapsulates the configuration space of 2 surfaces rolling along one another without slipping and twisting. The authors in [5] then found new examples of flat $(2,3,5)$-distributions that arise from rolling bodies, prompting further search in [8]. The solutions to (1.2) give examples of 4-dimensional split signature metrics that have their An–Nurowski twistor distributions having split G_2 as its group of symmetries and we exhibit them in Section 7. Let us recall some facts about Chazy's equation and its generalised version.

2 Chazy's equation

The study of Chazy's equation is a very rich subject and has received alot of attention because of its connection to other diverse fields such as integrable systems and modular forms. See for instance [2, 10, 11, 14]. We will review here some facts about Chazy's equation we need for the paper. Chazy [12, 13] studied the nonlinear 3rd order ODE

$$y'''(x) - 2y(x)y''(x) + 3(y'(x))^2 = 0 \tag{2.1}$$

in the context of investigating its Painlevé property. Solutions to equation (2.1) turn out to depend on hypergeometric functions. For further details, see [2] or [14]. Treat x as a dependent variable of s so that

$$x(s) = \frac{z_2(s)}{z_1(s)},$$

where $z_1(s)$, $z_2(s)$ are linearly independent solutions to the second order hypergeometric differential equation

$$s(1-s)z'' + (c - (a+b+1)s)z' - abz = 0. \tag{2.2}$$

Here a, b, c are constants to be determined. The general solution to this ODE (2.2) is given by hypergeometric functions

$$z(s) = \mu\, {}_2F_1(a,b;c;s) + \nu\, {}_2F_1(a-c+1, b-c+1; 2-c; s)s^{1-c}.$$

Here μ, ν are constants. A computation gives

$$\mathrm{d}x = \frac{z_1\dot{z}_2 - z_2\dot{z}_1}{(z_1)^2}\mathrm{d}s,$$

where dot denotes derivative with respect to s. We deduce that

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{(z_1)^2}{z_1\dot{z}_2 - z_2\dot{z}_1}\frac{\mathrm{d}}{\mathrm{d}s}.$$

Applying the derivative to Chazy's solution for y given by

$$y = 6\frac{\mathrm{d}}{\mathrm{d}x}\log z_1 = \frac{6z_1\dot{z}_1}{z_1\dot{z}_2 - z_2\dot{z}_1}, \tag{2.3}$$

we find that (2.1) is satisfied precisely when (a,b,c) is one of

$$\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}\right), \qquad \left(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}\right), \qquad \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right),$$

provided both a and b are non-zero. The equations (2.2) for the first two values of (a,b,c) are related by a linear transformation of the form $s \mapsto 1-s$, while the solutions for the second and third values are related by a quadratic transformation (see [23, equation (2)]) given by

$${}_2F_1\left(\frac{1}{6}, \frac{1}{6}; \frac{2}{3}; s\right) = {}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{2}{3}; 4s(1-s)\right).$$

The general solution to (2.1) thus depend on hypergeometric functions. If either one of a or b is zero (say $b=0$), solutions to (2.1) can be easily and explicitly described. The solutions to (2.2) with $b=0$ are given by

$$z(s) = \mu + \nu\, {}_2F_1(1-c, a+1-c; 2-c; s)s^{1-c} = \mu - \frac{\nu\pi(c-1)}{\sin(\pi c)}P_{c-1}^{(1-c,\ a-c)}(1-2s)s^{1-c},$$

where $P_n^{(a_1,b_1)}$ is the Jacobi polynomial. Taking $z_1(s) = \nu\, {}_2F_1(1-c, a+1-c; 2-c; s)s^{1-c}$ and $z_2(s) = \mu$, a computation shows that

$$x(s) = \frac{\mu}{\nu\, {}_2F_1(1-c, a+1-c; 2-c; s)s^{1-c}}$$

and

$$y(x(s)) = -6\frac{\nu}{\mu}\,{}_2F_1(1-c, a+1-c; 2-c; s)s^{1-c}.$$

Switching back to the original independent variable x, this gives

$$y(x) = -\frac{6}{x}$$

as one solution to (2.1). This solution is invariant under translations of the form $x \mapsto x + C$. In [12, 13], Chazy also observed that

$$y = -\frac{6}{x+C} - \frac{B}{(x+C)^2}$$

is a solution to (2.1). It is well-known that Chazy's equation and its generalised version can be rewritten as a first order system. This provides different parametrisations of y, in addition to the solution (2.3) originally given by Chazy. This will be discussed in Section 5. The method discussed here can also be applied to the generalised Chazy equation.

3 Generalised Chazy equations

The generalised Chazy equation is given by

$$y'''(x) - 2y''(x)y(x) + 3(y'(x))^2 - \frac{4}{36-k^2}(6y'(x) - y(x)^2)^2 = 0 \tag{3.1}$$

for $k \neq \pm 6$. We have the following:

Proposition 3.1. *Let* $x(s) = \frac{z_2(s)}{z_1(s)}$ *where* $z_1(s)$, $z_2(s)$ *are linearly independent solutions to the hypergeometric differential equation* (2.2) *where* (a, b, c) *is one of*

$$\left(\frac{k-6}{12k}, \frac{k+6}{12k}, \frac{1}{2}\right), \qquad \left(\frac{k-6}{6k}, \frac{k+6}{6k}, \frac{2}{3}\right), \qquad \left(\frac{k-6}{12k}, \frac{k+6}{12k}, \frac{2}{3}\right).$$

Then

$$y(x(s)) = 6\frac{\mathrm{d}}{\mathrm{d}x}\log z_1 = \frac{6z_1\dot{z}_1}{z_1\dot{z}_2 - z_2\dot{z}_1}$$

satisfies equation (3.1).

Proof. Analogous to solving Chazy's equation (2.1), we find that the generalised equation (3.1) holds provided

$$\begin{aligned}
&6ab(z_1)^8\Big(6((a-b)k - 6(a+b))((a-b)k + 6(a+b))s^2 \\
&\quad + \big((24ab - 12(a+b)c + 5(a+b) + (2c-1))k^2 + 432(a+b)c - 180(a+b) - 72c + 36\big)s \\
&\quad\quad + (k-6)(k+6)(2c-1)(3c-2)\Big) = 0.
\end{aligned}$$

For $a, b \neq 0$, solving the system of equations

$$\begin{aligned}
&6((a-b)k - 6(a+b))((a-b)k + 6(a+b)) = 0, \\
&(24ab - 12(a+b)c + 5(a+b) + (2c-1))k^2 + 432(a+b)c - 180(a+b) - 72c + 36 = 0, \\
&(k-6)(k+6)(2c-1)(3c-2) = 0,
\end{aligned}$$

gives the list of (a, b, c) as above. We exclude the case where $(a, b, c) = \left(0, 0, \frac{1}{2}\right)$. Note that interchanging a and b gives the same solution so that the full list is symmetric in a and b. ∎

When either a or b is zero, we again get $y(x) = -\frac{6}{x}$ as a solution. In [12, 13], Chazy noted that

$$y = \frac{k-6}{2(x+C)} - \frac{k+6}{2(x+B)} \tag{3.2}$$

is also a solution to (3.1). As a corollary to Proposition 3.1, we have

Corollary 3.2. *Let $q(s) = \frac{z_2(s)}{z_1(s)}$ where $z_1(s)$, $z_2(s)$ are linearly independent solutions to the hypergeometric differential equation (2.2) with (a,b,c) one of*

$$\left(-\frac{2}{3}, \frac{5}{6}, \frac{1}{2}\right), \qquad \left(-\frac{4}{3}, \frac{5}{3}, \frac{2}{3}\right), \qquad \left(-\frac{2}{3}, \frac{5}{6}, \frac{2}{3}\right).$$

Then

$$I(q(s)) = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1 = \frac{6z_1\dot{z}_1}{z_1\dot{z}_2 - z_2\dot{z}_1}$$

satisfies equation (1.2).

A Painlevé type analysis of equation (3.1) as done in [14] shows that the leading orders for analytic solutions to (3.1) occur at -6, $-3+\frac{k}{2}$ or $-3-\frac{k}{2}$. This corresponds to solutions of (3.1) given by

$$y(x) = -\frac{6}{x}, \qquad y(x) = \frac{-3+\frac{k}{2}}{x}, \qquad y(x) = \frac{-3-\frac{k}{2}}{x}.$$

These solutions are invariant under translations of the form $x \mapsto x + C$. In the case of $k = \pm\frac{2}{3}$ obtained in (1.2), we have

$$I(q) = -\frac{6}{q}, \qquad I(q) = -\frac{10}{3q}, \qquad I(q) = -\frac{8}{3q}.$$

Along with the zero solution $I(q) = 0$, these solutions correspond respectively (modulo constants of integration) to the well-known explicit solutions to (1.1):

$$F(q) = q^{-1}, \qquad F(q) = q^{\frac{1}{3}}, \qquad F(q) = q^{\frac{2}{3}}, \qquad F(q) = q^2.$$

For these functions of a single variable q the associated $(2,3,5)$-distributions have vanishing Cartan invariant and therefore have G_2 as their local symmetry.

4 Relationship to ODE studied by Dunajski and Sokolov

For the function $y = y(t)$, the 7^{th} order nonlinear ODE studied in [16] is given by

$$\begin{aligned} &10\big(y^{(3)}\big)^3 y^{(7)} - 70\big(y^{(3)}\big)^2 y^{(4)} y^{(6)} - 49\big(y^{(3)}\big)^2\big(y^{(5)}\big)^2 \\ &\qquad + 280\big(y^{(3)}\big)\big(y^{(4)}\big)^2 y^{(5)} - 175\big(y^{(4)}\big)^4 = 0. \end{aligned} \tag{4.1}$$

This is the unique 7^{th} order ODE admitting the submaximal contact symmetry group of dimension ten (see [16, 21]) and its relationship to equation (1.1) was originally explored in [6]. It is instructive to consider the 6^{th} order ODE (for the Legendre transformation later on):

$$\begin{aligned} &10\big(H^{(2)}\big)^3 H^{(6)} - 70\big(H^{(2)}\big)^2 H^{(3)} H^{(5)} - 49\big(H^{(2)}\big)^2\big(H^{(4)}\big)^2 \\ &\qquad + 280\big(H^{(2)}\big)\big(H^{(3)}\big)^2 H^{(4)} - 175\big(H^{(3)}\big)^4 = 0 \end{aligned} \tag{4.2}$$

with $H(t) = y'(t)$. Let us show that this ODE can be reduced to a generalised Chazy equation. Again working locally in an open set where $y'''(t)$ is non-zero, and assuming $y'''(t)$ to be positive, we can make the substitution $\mathrm{e}^{p(t)} = y^{(3)}$ to get

$$\mathrm{e}^{p(t)}\big(10p^{(4)} - 30p'p''' - 19(p'')^2 + 32(p')^2p'' - 4(p')^4\big) = 0. \tag{4.3}$$

We note that this 4^{th} order ODE historically appears in [9, Section XII, formula (12)], where it first arises as the obstruction to integrability for $(2,3,5)$-distributions of the form $\mathcal{D}_{F(q)}$. This will be made clear below once we show that (4.2) is the Legendre transform of (1.1) [6] and we will discuss this further in Section 6. Thus, for $v(t) = p'(t)$, we obtain the third order ODE

$$10v''' - 30vv'' - 19(v')^2 + 32v'v^2 - 4v^4 = 0. \tag{4.4}$$

Rescaling $v(t)$ by $u(t) = \frac{3}{2}v(t)$, we put (4.4) into the normal form

$$u''' - 2u''u + 3(u')^2 - \frac{4}{36 - \left(\frac{3}{2}\right)^2}\big(6u' - u^2\big)^2 = 0. \tag{4.5}$$

We therefore see that the ODE that Dunajski and Sokolov study in [16] reduces to a generalised Chazy equation (4.5) with parameter $k' = \pm\frac{3}{2}$, related to the generalised Chazy equation (1.2) just by taking the reciprocals $(k')^2 = \frac{1}{k^2}$ of the corresponding parameters.

Let $t(s) = \frac{w_2(s)}{w_1(s)}$ where $w_1(s)$, $w_2(s)$ are linearly independent solutions to the hypergeometric differential equation (2.2) with (a, b, c) one of

$$\left(-\frac{1}{4}, \frac{5}{12}, \frac{1}{2}\right), \qquad \left(-\frac{1}{4}, \frac{5}{12}, \frac{2}{3}\right), \qquad \left(-\frac{1}{2}, \frac{5}{6}, \frac{2}{3}\right).$$

The solution to (4.5) is then given by $u = 6\frac{\mathrm{d}}{\mathrm{d}t}\log w_1$. A similar leading order analysis as before shows that the leading orders occur at

$$-6, \quad -\frac{9}{8}, \quad -\frac{15}{8}.$$

This corresponds to solutions of (4.5) given by

$$u(t) = -\frac{6}{t}, \qquad u(t) = -\frac{9}{4t}, \qquad u(t) = -\frac{15}{4t}.$$

Along with the zero solution $u(t) = 0$, these correspond respectively (modulo constants of integration) to solutions of (4.2) given by

$$H(t) = t^{-2}, \qquad H(t) = t^{\frac{1}{2}}, \qquad H(t) = t^{-\frac{1}{2}}, \qquad H(t) = t^2.$$

In [6, Proposition 2.2], it is shown that a Legendre transformation takes (1.1) to (4.1). Hence we may hypothesise that amongst all 3^{rd} order generalised Chazy equations, only those with the parameters $k' = \pm\frac{3}{2}$, $k = \pm\frac{2}{3}$ have in addition solutions that can be obtained from the dual equation via a Legendre transform.

Proposition 4.1 ([6, Proposition 2.2])**.** *Consider the Legendre transformation*

$$F(q) + H(t) = qt.$$

Then $F(q)$ satisfies the ODE (1.1) iff $H(t)$ satisfies the ODE (4.2).

Proof. Applying the exterior derivative to the relation gives

$$(F' - t)\mathrm{d}q + (H' - q)\mathrm{d}t = 0,$$

so that we take $F' = t$, $H' = q$ and applying

$$\frac{\mathrm{d}}{\mathrm{d}q} = \frac{1}{H''}\frac{\mathrm{d}}{\mathrm{d}t}$$

we obtain $F'' = \frac{1}{H''}$, $F^{(3)} = -\frac{H^{(3)}}{(H'')^3}$, etc. A computation shows that the 6^{th} order ODE (1.1) holds for F iff (4.2) holds for H. ■

In light of the solutions obtained by solving the generalised Chazy equations, we can pass to

$$F(q) = \iint \mathrm{e}^{\int \frac{I(q)}{2}\mathrm{d}q}\mathrm{d}q\mathrm{d}q, \tag{4.6}$$

where $q = \frac{z_2(s)}{z_1(s)}$ and $I(q) = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1$ are given in Corollary 3.2. This gives

$$F(q) = \iint (z_1)^3\mathrm{d}q\mathrm{d}q.$$

Similarly, for the dual equation (4.2) under the Legendre transform we pass to

$$H(t) = \iint \mathrm{e}^{\int \frac{2u(t)}{3}\mathrm{d}t}\mathrm{d}t\mathrm{d}t,$$

where $t = \frac{w_2(s)}{w_1(s)}$ and $u(t) = 6\frac{\mathrm{d}}{\mathrm{d}t}\log w_1$ are solutions to (4.5). This gives

$$H(t) = \iint (w_1)^4\mathrm{d}t\mathrm{d}t.$$

We have

Lemma 4.2. *There exists a Legendre transformation between Chazy's solutions of* (1.2) *and* (4.5) *given by taking*

$$w_1(s) = z_1^{-\frac{3}{4}}, \qquad w_2(s) = (z_1)^{-\frac{3}{4}}\int (z_1)(\dot{z}_2 z_1 - \dot{z}_1 z_2)\mathrm{d}s.$$

This defines a mapping

$$q = \frac{z_2(s)}{z_1(s)} \mapsto t = \frac{w_2(s)}{w_1(s)} = \int z_1(\dot{z}_2 z_1 - \dot{z}_1 z_2)\mathrm{d}s.$$

If $I(q) = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1$ *solves* (1.2) *where* $z_1(s)$ *and* $z_2(s)$ *are given in Corollary* 3.2*, then* $u(t) = 6\frac{\mathrm{d}}{\mathrm{d}t}\log w_1$ *solves the dual ODE* (4.5)*. Consequently, if* $F(q) = \iint (z_1)^3\mathrm{d}q\mathrm{d}q$ *solves* (1.1)*, then*

$$H(t) = \iint (w_1)^4\mathrm{d}t\mathrm{d}t = \iint (z_1)^{-2}(\dot{z}_2 z_1 - \dot{z}_1 z_2)\mathrm{d}s\, z_1(\dot{z}_2 z_1 - \dot{z}_1 z_2)\mathrm{d}s$$

solves the 6^{th} *order ODE* (4.2)*. For the converse, the Legendre transform is given by*

$$z_1(s) = w_1^{-\frac{4}{3}}, \qquad z_2(s) = (w_1)^{-\frac{4}{3}}\int (w_1)^2(\dot{w}_2 w_1 - \dot{w}_1 w_2)\mathrm{d}s.$$

This sends

$$t = \frac{w_2(s)}{w_1(s)} \mapsto q = \frac{z_2(s)}{z_1(s)} = \int (w_1)^2(\dot{w}_2 w_1 - \dot{w}_1 w_2)\mathrm{d}s.$$

In particular, if $u(t) = 6\frac{\mathrm{d}}{\mathrm{d}t}\log w_1$ *solves the dual ODE* (4.5)*, then* $I(q) = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1$ *solves* (1.2)*. Hence, if* $H(t) = \iint (w_1)^4\mathrm{d}t\mathrm{d}t$ *solves the* 6^{th} *order ODE* (4.2)*, then*

$$F(q) = \iint (z_1)^3\mathrm{d}q\mathrm{d}q = \iint (w_1)^{-2}(\dot{w}_2 w_1 - \dot{w}_1 w_2)\mathrm{d}s\,(w_1)^2(\dot{w}_2 w_1 - \dot{w}_1 w_2)\mathrm{d}s$$

solves (1.1)*.*

Proof. We observe that as a consequence of Chazy's solutions, the Legendre transform in Proposition 4.1 gives

$$\frac{w_2}{w_1} = t = F' = \int (z_1)^3\mathrm{d}q = \int z_1(\dot{z}_2 z_1 - \dot{z}_1 z_2)\mathrm{d}s,$$
$$\frac{z_2}{z_1} = q = H' = \int (w_1)^4\mathrm{d}t = \int (w_1)^2(\dot{w}_2 w_1 - \dot{w}_1 w_2)\mathrm{d}s,$$

and therefore

$$\frac{\dot{w}_2 w_1 - \dot{w}_1 w_2}{(w_1)^2} = z_1(\dot{z}_2 z_1 - \dot{z}_1 z_2),$$
$$\frac{\dot{z}_2 z_1 - \dot{z}_1 z_2}{(z_1)^2} = (w_1)^2(\dot{w}_2 w_1 - \dot{w}_1 w_2).$$

Together this yields $(z_1)^3 = (w_1)^{-4}$, from which we deduce

$$w_1 = z_1^{-\frac{3}{4}} \qquad \text{and} \qquad w_2 = w_1\int z_1(\dot{z}_2 z_1 - \dot{z}_1 z_2)\mathrm{d}s = z_1^{-\frac{3}{4}}\int z_1(\dot{z}_2 z_1 - \dot{z}_1 z_2)\mathrm{d}s.$$

For the converse, we find

$$z_1 = w_1^{-\frac{4}{3}} \qquad \text{and} \qquad z_2 = z_1\int (w_1)^2(\dot{w}_2 w_1 - \dot{w}_1 w_2)\mathrm{d}s = w_1^{-\frac{4}{3}}\int (w_1)^2(\dot{w}_2 w_1 - \dot{w}_1 w_2)\mathrm{d}s.$$

The rest follows from a routine computation. ■

In [16, formula (8)], a family of solutions to (4.1) is found to be given by the algebraic curve

$$(y + f(t))^2 = (t-a)(t-b)^3,$$

with $a \neq b$, and $f(t)$ a quadratic. This gives

$$y = \pm\sqrt{(t-a)(t-b)^3} - f(t).$$

We obtain a solution to (4.1) with

$$y^{(3)} = \pm\frac{3}{8}\frac{(t-b)^6(a-b)^3}{(y+f)^5} = \pm\frac{3(a-b)^3}{8(t-a)^2(y+f)}.$$

We find that for this solution, it yields

$$u(t) = \frac{3}{4}\frac{5b + 3a - 8t}{(t-a)(t-b)} = -\frac{15}{4(t-a)} - \frac{9}{4(t-b)}$$

as a solution to the generalized Chazy's equation with parameter $k^2 = \frac{9}{4}$. This corresponds to the solution given by Chazy in (3.2). It will be interesting to determine the solutions of (4.5) from the general solution given by [16, formula (13)].

5 First order system and different parametrisations of Chazy's equations

In this section, we first show that the generalised Chazy equation is equivalent to solving a third order differential equation involving the Schwarzian derivative and a potential term $V(s)$. It is well-known that solutions to the generalised Chazy equation (3.1) can be rewritten as a first order system. For further details, see [2]. The first order system provides different parametrisations of the solutions, in addition to the one given by (2.3). We compute the solutions to the generalised Chazy equation (1.2) with $k = \pm\frac{2}{3}$ for the different parametrisations below and present them in Tables 1, 2 and 3. We also show how these solutions are related to one another by algebraic transformation of hypergeometric functions.

Let Ω_1, Ω_2, Ω_3 be functions of q. Let dot denote differentiation with respect to q. Then consider

$$\begin{aligned}
\dot{\Omega}_1 &= \Omega_2\Omega_3 - \Omega_1(\Omega_2+\Omega_3) + \tau^2,\\
\dot{\Omega}_2 &= \Omega_3\Omega_1 - \Omega_2(\Omega_3+\Omega_1) + \tau^2,\\
\dot{\Omega}_3 &= \Omega_1\Omega_2 - \Omega_3(\Omega_1+\Omega_2) + \tau^2,
\end{aligned} \tag{5.1}$$

where

$$\tau^2 = \alpha^2(\Omega_1-\Omega_2)(\Omega_3-\Omega_1) + \beta^2(\Omega_2-\Omega_3)(\Omega_1-\Omega_2) + \gamma^2(\Omega_3-\Omega_1)(\Omega_2-\Omega_3)$$

and α, β, γ are constants. Introducing the parameter

$$s(q) = \frac{\Omega_1-\Omega_3}{\Omega_2-\Omega_3},$$

we find that

$$\Omega_1 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}}{s(s-1)}, \qquad \Omega_2 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}}{s-1}, \qquad \Omega_3 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}}{s}.$$

The system of equations (5.1) are satisfied iff

$$\{s,q\} + \frac{\dot{s}^2}{2}V(s) = 0, \tag{5.2}$$

where

$$\{s,q\} = \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2$$

is the Schwarzian derivative of $s(q)$ and

$$V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}.$$

Switching independent and dependent variables in (5.2), we have

$$\{s,q\} = -(\dot{s})^2\{q,s\},$$

so that the dual of (5.2) is

$$\{q,s\} - \frac{1}{2}V(s) = 0.$$

The general solution is given by

$$q(s) = \frac{u_2(s)}{u_1(s)}$$

where u_1, u_2 are linearly independent solutions of the second order ODE

$$u'' + \frac{1}{4}V(s)u = 0. \tag{5.3}$$

The general solution of (5.3) suggests taking

$$u(s) = (s-1)^{\frac{1-\gamma}{2}} s^{\frac{1-\beta}{2}} z(s)$$

(cf. [3]) to transform (5.3) into the hypergeometric differential equation (2.2) with

$$a = \frac{1}{2}(1-\alpha-\beta-\gamma), \qquad b = \frac{1}{2}(1+\alpha-\beta-\gamma), \qquad c = 1-\beta.$$

In [2], it was determined that taking

$$y = -2(\Omega_1+\Omega_2+\Omega_3) = \frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}^3}{s^2(s-1)^2} \tag{5.4}$$

gives solutions to the generalised Chazy equation (3.1) with variable q whenever $\alpha=\beta=\gamma=\frac{2}{k}$ or $\alpha=\frac{2}{k}$, $\beta=\gamma=\frac{1}{3}$ and its cyclic permutations. For $\alpha=\beta=\gamma=\frac{2}{k}$ this gives $(a,b,c) = \left(\frac{k-6}{2k}, \frac{k-2}{2k}, \frac{k-2}{k}\right)$. For $\alpha=\frac{2}{k}$, $\beta=\gamma=\frac{1}{3}$ with cyclic permutations, this gives respectively

$$(a,b,c) = \left(\frac{k-6}{6k}, \frac{k+6}{6k}, \frac{2}{3}\right), \qquad \left(\frac{k-6}{6k}, \frac{k-2}{2k}, \frac{k-2}{k}\right), \qquad \left(\frac{k-6}{6k}, \frac{k-2}{2k}, \frac{2}{3}\right),$$

with only the first coinciding with the list in Proposition 3.1. This suggests that the solutions to (5.2) are more general than the solution of the form $y = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1$ given by Chazy [13]. We can express Chazy's solution in terms of $s(q)$ as follows. A computation of the Wronskian of linearly independent solutions z_1, z_2 to (2.2) gives

$$W(z_1,z_2) = z_1\dot{z}_2 - z_2\dot{z}_1 = w_0(s-1)^{c-a-b-1}s^{-c}$$

for some non-zero constant w_0. The latter equality holds by solving the first order differential equation

$$W' = -\frac{c-(a+b+1)s}{s(1-s)}W.$$

See for instance [4]. From $q(s) = \frac{z_2(s)}{z_1(s)}$, we find that $\frac{\mathrm{d}}{\mathrm{d}q} = \frac{(z_1)^2}{z_1\dot{z}_2 - z_2\dot{z}_1}\frac{\mathrm{d}}{\mathrm{d}s}$. Applying this derivative to $s(q)$, we obtain

$$s'(q) = \frac{\mathrm{d}}{\mathrm{d}q}s(q) = \frac{(z_1)^2}{z_1\dot{z}_2 - z_2\dot{z}_1} = \frac{(z_1)^2}{W(z_1,z_2)}.$$

Hence

$$\begin{aligned}
s''(q) = \frac{\mathrm{d}}{\mathrm{d}q}s'(q) = s'(q)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{(z_1)^2}{W(z_1,z_2)}\right) &= s'(q)\left(\frac{2z_1\dot{z}_1}{W(z_1,z_2)} - \frac{(z_1)^2W'(z_1,z_2)}{W(z_1,z_2)^2}\right)\\
&= s'(q)\left(2s'(q)\frac{\dot{z}_1}{z_1} - s'(q)\frac{W'(z_1,z_2)}{W(z_1,z_2)}\right)\\
&= s'(q)\left(2s'(q)\frac{\dot{z}_1}{z_1} + s'(q)\frac{c-(a+b+1)s(q)}{s(q)(1-s(q))}\right)\\
&= s'(q)\left(2s'(q)\frac{\dot{z}_1}{z_1} + s'(q)\frac{c(1-s(q))-(a+b+1-c)s(q)}{s(q)(1-s(q))}\right),
\end{aligned}$$

and we get

$$\frac{\ddot{s}}{\dot{s}} = 2\dot{s}\frac{\dot{z}_1}{z_1} + c\frac{\dot{s}}{s} - (a+b+1-c)\frac{\dot{s}}{1-s}.$$

Therefore we have

$$\begin{aligned} y = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1 = 6\frac{(z_1)^2}{W(z_1,z_2)}\frac{\dot{z}_1}{z_1} = 6\dot{s}\frac{\dot{z}_1}{z_1} &= 3\frac{\ddot{s}}{\dot{s}} - 3c\frac{\dot{s}}{s} + 3(a+b+1-c)\frac{\dot{s}}{1-s} \\ &= 3\frac{\mathrm{d}}{\mathrm{d}q}\log\dot{s} - 3c\frac{\mathrm{d}}{\mathrm{d}q}\log s - 3(a+b+1-c)\frac{\mathrm{d}}{\mathrm{d}q}\log(1-s) \\ &= 3\frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}}{s^c(1-s)^{a+b+1-c}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}^6}{s^{6c}(1-s)^{6(a+b+1-c)}}. \end{aligned}$$

A comparison of Chazy's formula for $y = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1$ with the formula for y in (5.4) suggests taking $c = \frac{2}{3}$, $a+b = \frac{1}{3}$. This is satisfied by $(a,b,c) = \left(\frac{k-6}{6k}, \frac{k+6}{6k}, \frac{2}{3}\right)$ in Proposition 3.1.

For $k = \frac{2}{3}$ as in (1.2), we get the following solutions for u given in Table 1. For this Chazy parameter, the hypergometric series truncate and the solutions to (5.3) can be given by elementary functions. We have

Table 1.

(α,β,γ)	(a,b,c)	General solution to $u'' + \frac{1}{4}V(s)u = 0$
$(3,3,3)$	$(-4,-1,-2)$	$c_1\frac{2s-1}{s(s-1)} + c_2\frac{s^2(s-2)}{s-1}$
$\left(3,\frac{1}{3},\frac{1}{3}\right)$	$\left(-\frac{4}{3},\frac{5}{3},\frac{2}{3}\right)$	$c_1(3s-2)s^{\frac{2}{3}}(s-1)^{\frac{1}{3}} + c_2(3s-1)s^{\frac{1}{3}}(s-1)^{\frac{2}{3}}$
$\left(\frac{1}{3},3,\frac{1}{3}\right)$	$\left(-\frac{4}{3},-1,-2\right)$	$c_1\frac{2s-3}{s}(s-1)^{\frac{1}{3}} + c_2\frac{s-3}{s}(s-1)^{\frac{2}{3}}$
$\left(\frac{1}{3},\frac{1}{3},3\right)$	$\left(-\frac{4}{3},-1,\frac{2}{3}\right)$	$c_1\frac{2s+1}{s-1}s^{\frac{1}{3}} + c_2\frac{s+2}{s-1}s^{\frac{2}{3}}$

$$\begin{aligned} &{}_2F_1(-4,-1;-2;s) = 1-2s, &&{}_2F_1\left(-\frac{4}{3},\frac{5}{3};\frac{2}{3};s\right) = \frac{3s^2-4s+1}{(1-s)^{\frac{2}{3}}}, \\ &{}_2F_1\left(-\frac{4}{3},-1;-2;s\right) = 1-\frac{2}{3}s, &&{}_2F_1\left(-\frac{4}{3},-1;\frac{2}{3};s\right) = 1+2s. \end{aligned}$$

Moreover, the solutions in each row are related to one another by algebraic transformations of hypergeometric functions. The solutions in the second, third and fourth rows of Table 1 can be obtained from the first by a cubic transformation of hypergeometric functions (see [23, formula (23)]). Explicitly, to show how the solution in the third row is related to the first, let ω be a cube root of unity (solution to $\omega^2+\omega+1=0$) and consider the map

$$s \mapsto t = \frac{3(2\omega+1)s(s-1)}{(s+\omega)^3}$$

and the relation

$$\tilde{z}(t) = (1+\omega s)^{-4}z(s).$$

Then $\tilde{z}(t)$ is a solution to the hypergeometric differential equation (2.2) with $(a,b,c) = \left(-\frac{4}{3},-1,-2\right)$ iff $z(s)$ solves (2.2) with $(a,b,c) = (-4,-1,-2)$. The solutions in the last three rows are related to one another by fractional linear transformations. The symmetry $s \mapsto 1-s$ interchanges β and γ in $V(s)$ and hence transforms the solution in the 3rd row to the solution

given in the 4th row while $s \mapsto t = \frac{s}{s-1}$ and the relation $\tilde{z}(t) = (1-s)^{-\frac{4}{3}} z(s)$ transforms the solution in the 4th row to those in the 2nd.

A different parametrisation of Chazy's equations (cf. [10]) is also given by

$$y = -\Omega_1 - 2\Omega_2 - 3\Omega_3 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}^6}{s^4(s-1)^3}. \tag{5.5}$$

A comparison with Chazy's formula (2.3) yields $(a,b,c) = \left(\frac{k-6}{12k}, \frac{k+6}{12k}, \frac{2}{3}\right)$ from Proposition 3.1. The solution of the form (5.5) solves the generalised Chazy equation (3.1) whenever (α,β,γ) in (5.2) is given by

$$\left(\frac{1}{k}, \frac{1}{3}, \frac{1}{2}\right), \qquad \left(\frac{1}{k}, \frac{2}{k}, \frac{1}{2}\right) \qquad \text{or} \qquad \left(\frac{1}{k}, \frac{1}{3}, \frac{3}{k}\right).$$

The solution to (5.2) with $(\alpha,\beta,\gamma) = \left(\frac{1}{k}, \frac{1}{3}, \frac{1}{2}\right)$ is given by the Schwarz function J (see [2, 3]).

Considering the symmetry $s = 1-K$ brings (5.5) to

$$y = -\Omega_1 - 3\Omega_2 - 2\Omega_3 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}q}\log\frac{-\dot{K}^6}{(1-K)^4K^3} \tag{5.6}$$

and comparing with Chazy's formula gives $(a,b,c) = \left(\frac{k-6}{12k}, \frac{k+6}{12k}, \frac{1}{2}\right)$ from Proposition 3.1. The solution of the form (5.6) solves (3.1) whenever (α,β,γ) in (5.2) is given by

$$\left(\frac{1}{k}, \frac{1}{2}, \frac{1}{3}\right), \qquad \left(\frac{1}{k}, \frac{1}{2}, \frac{2}{k}\right) \qquad \text{or} \qquad \left(\frac{1}{k}, \frac{3}{k}, \frac{1}{3}\right).$$

The symmetry $s = 1-K$ permutes β and γ in the formula for $V(s)$ in (5.2). For $k = \frac{2}{3}$, the solutions to (5.3) with the parametrisation by (5.5) are presented in Table 2.

Table 2.

(α,β,γ)	(a,b,c)	General solution to $u'' + \frac{1}{4}V(s)u = 0$
$\left(\frac{3}{2}, \frac{1}{3}, \frac{1}{2}\right)$	$\left(\frac{5}{6}, -\frac{2}{3}, \frac{2}{3}\right)$	$c_1(s-1)^{\frac{1}{4}}s^{\frac{1}{2}}P_1^{\frac{1}{3}}(\sqrt{1-s}) + c_2(s-1)^{\frac{1}{4}}s^{\frac{1}{2}}Q_1^{\frac{1}{3}}(\sqrt{1-s})$
$\left(\frac{3}{2}, 3, \frac{1}{2}\right)$	$\left(-\frac{1}{2}, -2, -2\right)$	$c_1\frac{(s-1)^{\frac{3}{4}}}{s} + c_2\frac{(s-1)^{\frac{1}{4}}}{s}(s^2+4s-8)$
$\left(\frac{3}{2}, \frac{1}{3}, \frac{9}{2}\right)$	$\left(-\frac{7}{6}, -\frac{8}{3}, \frac{2}{3}\right)$	$c_1 s^{\frac{2}{3}}(s-1)^{\frac{11}{4}}\,{}_2F_1\left(\frac{13}{6}, \frac{11}{3}; \frac{4}{3}; s\right) + c_2 s^{\frac{1}{3}}(s-1)^{\frac{11}{4}}\,{}_2F_1\left(\frac{11}{6}, \frac{10}{3}; \frac{2}{3}; s\right)$

We also note that we have

$$\begin{aligned}
{}_2F_1\left(\frac{5}{6}, -\frac{2}{3}; \frac{2}{3}; s\right) &= \frac{\Gamma\left(\frac{2}{3}\right)^3\sqrt{3}}{\pi}P_{\frac{2}{3}}^{\left(-\frac{1}{3},-\frac{1}{2}\right)}(1-2s),\\
{}_2F_1\left(-\frac{1}{2}, -2; -2; s\right) &= \frac{1}{8}(8-4s-s^2),\\
{}_2F_1\left(-\frac{7}{6}, -\frac{8}{3}; \frac{2}{3}; s\right) &= \frac{10}{7}\frac{\Gamma\left(\frac{2}{3}\right)^3\sqrt{3}}{\pi}P_{\frac{8}{3}}^{\left(-\frac{1}{3},-\frac{9}{2}\right)}(1-2s).
\end{aligned}$$

The algebraic transformations relating the solutions between the rows are given as follows. The map that takes the solution from the second row to the solution in the first row of Table 2 is a composition of fractional linear transformations and cubic transformation due to Goursat (see [23, formulas (20)–(22)] and [17, formula (123)]). To see this, we first consider the map

$$s \mapsto t = \frac{s(9-s)^2}{(s+3)^3}$$

and the relation

$$\tilde{z}(t) = \left(1 + \frac{s}{3}\right)^{-2} z(s).$$

Then $\tilde{z}(t)$ satisfies (2.2) with $(a,b,c) = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{1}{2}\right)$ iff $z(s)$ satisfies (2.2) with $(a,b,c) = \left(-2, -\frac{1}{2}, \frac{1}{2}\right)$. Now

$${}_2F_1\left(-\frac{2}{3}, -\frac{1}{3}; \frac{1}{2}; t\right) = (1-t)^{\frac{2}{3}}\, {}_2F_1\left(-\frac{2}{3}, \frac{5}{6}; \frac{1}{2}; \frac{t}{t-1}\right)$$

while the map $s \mapsto 1-s$ takes the differential equation (2.2) with $(a,b,c) = \left(-\frac{2}{3}, \frac{5}{6}, \frac{1}{2}\right)$ to $(a,b,c) = \left(-\frac{2}{3}, \frac{5}{6}, \frac{2}{3}\right)$. Moreover, we have

$${}_2F_1\left(-2, -\frac{1}{2}; \frac{1}{2}; 1-s\right) = \frac{1}{3}\left(8 - 4s - s^2\right) = \frac{8}{3}\, {}_2F_1\left(-2, -\frac{1}{2}; -2; s\right),$$

so that all together the composition gives the map

$$s \mapsto \tilde{t} = -\frac{(s-4)^3}{27s^2}$$

and the relation

$$\tilde{z}(\tilde{t}) = s^{-\frac{4}{3}} z(s).$$

Thus $\tilde{z}(\tilde{t})$ satisfies (2.2) with $(a,b,c) = \left(-\frac{2}{3}, \frac{5}{6}, \frac{2}{3}\right)$ iff $z(s)$ satisfies (2.2) with $(a,b,c) = \left(-2, -\frac{1}{2}, -2\right)$.

To obtain the solution given in the third row from those in the first row requires a transformation of degree 4 (see [23, formulas (25)–(27)] and [17, equation (131)]). Consider the map

$$s \mapsto t = -\frac{s(s+8)^3}{64(1-s)^3}$$

and the relation

$$\tilde{z}(t) = (1-s)^{\frac{5}{2}} z(s).$$

Then $\tilde{z}(t)$ satisfies (2.2) with $(a,b,c) = \left(\frac{5}{6}, -\frac{2}{3}, \frac{2}{3}\right)$ iff $z(s)$ satisfies (2.2) with $(a,b,c) = \left(\frac{11}{6}, \frac{10}{3}, \frac{2}{3}\right)$. Finally, we use the Euler transformation,

$${}_2F_1\left(-\frac{7}{6}, -\frac{8}{3}; \frac{2}{3}; s\right) = (1-s)^{\frac{9}{2}}\, {}_2F_1\left(\frac{11}{6}, \frac{10}{3}; \frac{2}{3}; s\right)$$

to obtain the solution in the 3$^{\text{rd}}$ row.

There is furthermore a degree 6 transformation relating the solution in the first row of Table 1 to the solution in the first row of Table 2 (cf. [23, equation (28)], [17, equation (134)] and [1, Table 1]). Consider the map

$$s \mapsto t = \frac{27s^2(s-1)^2}{4(s^2-s+1)^3}$$

and the relation

$$\tilde{z}(t) = (1 - s + s^2)^{-2} z(s).$$

Then $\tilde{z}(t)$ satisfies (2.2) with $(a,b,c) = \left(-\frac{1}{3}, -\frac{2}{3}, -\frac{1}{2}\right)$ iff $z(s)$ satisfies (2.2) with $(a,b,c) = (-4,-1,-2)$. Furthermore $s \mapsto 1-s$ takes the solution to (2.2) with $(a,b,c) = \left(-\frac{1}{3}, -\frac{2}{3}, -\frac{1}{2}\right)$ to the solution with $(a,b,c) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{1}{2}\right)$ and an Euler transformation followed by $s \mapsto 1-s$ again takes this to the solution with $(a,b,c) = \left(\frac{5}{6}, -\frac{2}{3}, \frac{2}{3}\right)$ as given in row 1 of Table 2.

Finally, consider the parametrisation given by

$$y = -4\Omega_1 - \Omega_2 - \Omega_3 = \frac{\mathrm{d}}{\mathrm{d}q} \log \frac{\dot{s}^3}{(s-1)^{\frac{5}{2}} s^{\frac{5}{2}}}. \tag{5.7}$$

We find that (α,β,γ) is one of

$$\left(\frac{4}{k}, \frac{1}{k}, \frac{1}{k}\right), \qquad \left(\frac{2}{3}, \frac{1}{k}, \frac{1}{k}\right), \qquad \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right).$$

For $(\alpha,\beta,\gamma) = \left(\frac{4}{k}, \frac{1}{k}, \frac{1}{k}\right)$, we find that $(a,b,c) = \left(\frac{k-6}{2k}, \frac{k+2}{2k}, \frac{k-1}{k}\right)$. For $k = \frac{2}{3}$, we obtain $(a,b,c) = (-4, 2, -\frac{1}{2})$. Let us relate the solution to the differential equation (2.2) with $(a,b,c) = (-4,2,-\frac{1}{2})$ to the solution given in the second row of Table 2. For $(\alpha,\beta,\gamma) = \left(\frac{1}{2}, \frac{1}{k}, \frac{2}{k}\right)$, we find that $(a,b,c) = \left(\frac{k-6}{4k}, \frac{3k-6}{4k}, \frac{k-1}{k}\right)$. We have

$$\begin{aligned}
{}_2F_1\left(\frac{1}{2} - \frac{3}{k}, \frac{1}{2} + \frac{1}{k}; 1 - \frac{1}{k}; s\right) &= {}_2F_1\left(\frac{1}{4} - \frac{3}{2k}, \frac{1}{4} + \frac{1}{2k}; 1 - \frac{1}{k}; 4s(1-s)\right) \\
&= (1 - 4s(1-s))^{-\frac{1}{4} + \frac{3}{2k}} {}_2F_1\left(\frac{1}{4} - \frac{3}{2k}, \frac{3}{4} - \frac{3}{2k}; 1 - \frac{1}{k}; \frac{4s(1-s)}{4s(1-s)-1}\right)
\end{aligned}$$

and thus for $k = \frac{2}{3}$, we have

$${}_2F_1\left(-4, 2; -\frac{1}{2}; s\right) = (1 - 4s(1-s))^2 \, {}_2F_1\left(-2, -\frac{3}{2}; -\frac{1}{2}; \frac{4s(1-s)}{4s(1-s)-1}\right).$$

Again $s \mapsto 1-s$ brings the solution to (2.2) with $(a,b,c) = \left(-2, -\frac{3}{2}, -\frac{1}{2}\right)$ to the solution with $(a,b,c) = \left(-2, -\frac{3}{2}, -2\right)$ and an Euler transform gives the solution with $(a,b,c) = \left(-2, -\frac{1}{2}, -2\right)$.

For $(\alpha,\beta,\gamma) = \left(\frac{2}{3}, \frac{1}{k}, \frac{1}{k}\right)$, we find that $(a,b,c) = \left(\frac{k-6}{6k}, \frac{5k-6}{6k}, \frac{k-1}{k}\right)$. For $k = \frac{2}{3}$, this gives $(a,b,c) = \left(-\frac{4}{3}, -\frac{2}{3}, -\frac{1}{2}\right)$. We have a degree 2 transformation of the solution to (2.2) with this value to the solution in the first row of Table 2 given by the map

$$s \mapsto t = -\frac{1}{4s(s-1)}$$

and the relation

$$\tilde{z}(t) = \frac{1}{4}(s(1-s))^{-\frac{2}{3}} z(s).$$

We have $\tilde{z}(t)$ satisfying (2.2) with $(a,b,c) = \left(-\frac{2}{3}, \frac{5}{6}, \frac{2}{3}\right)$ iff $z(s)$ satisfies (2.2) with $(a,b,c) = \left(-\frac{4}{3}, -\frac{2}{3}, -\frac{1}{2}\right)$. Let us summarise the solutions to (5.3) with parametrisation given by (5.7) in Table 3.

We also note that we have

$$\begin{aligned}
{}_2F_1\left(-4, 2; -\frac{1}{2}; s\right) &= -\frac{128}{5} P_4^{\left(-\frac{3}{2}, -\frac{3}{2}\right)}(1-2s) = -128s^4 + 256s^3 - 144s^2 + 16s + 1, \\
{}_2F_1\left(-\frac{4}{3}, -\frac{2}{3}; -\frac{1}{2}; s\right) &= -\frac{16}{27} \frac{\pi^2 2^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)^3} P_{\frac{4}{3}}^{\left(-\frac{3}{2}, -\frac{3}{2}\right)}(1-2s), \\
{}_2F_1\left(0, \frac{2}{3}; \frac{5}{6}; s\right) &= 1.
\end{aligned}$$

Table 3.

(α,β,γ)	(a,b,c)	General solution to $u''+\frac{1}{4}V(s)u=0$
$\left(6,\frac{3}{2},\frac{3}{2}\right)$	$\left(-4,2,-\frac{1}{2}\right)$	$c_1(2s-1)(s(s-1))^{\frac{5}{4}}+c_2\frac{128s^4-256s^3+144s^2-16s-1}{(s(s-1))^{\frac{1}{4}}}$
$\left(\frac{2}{3},\frac{3}{2},\frac{3}{2}\right)$	$\left(-\frac{4}{3},-\frac{2}{3},-\frac{1}{2}\right)$	$c_1(s(s-1))^{\frac{5}{4}}\,{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};s\right)+c_2\frac{(s-1)^{\frac{5}{4}}}{s^{\frac{1}{4}}}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};-\frac{1}{2};s\right)$
$\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right)$	$\left(0,\frac{2}{3},\frac{5}{6}\right)$	$c_1(s(s-1))^{\frac{7}{12}}\,{}_2F_1\left(\frac{1}{3},1;\frac{7}{6};s\right)+c_2s^{\frac{5}{12}}(1-s)^{\frac{5}{12}}$

Comparing the parametrisation (5.7) with solutions of the form (2.3), we obtain $(a,b,c)=\left(0,\frac{2}{3},\frac{5}{6}\right)$. For this solution given in the third row, we obtain solutions of the form $y(q)=-\frac{6}{q}$. Up to fractional linear transformations in the variable s, we have 7 classes of solutions to (1.2) determined by the solutions to (5.2). They are given by the solutions with (α,β,γ) either $(3,3,3)$ or $\left(3,\frac{1}{3},\frac{1}{3}\right)$ for $I(q)$ given by (5.4), (α,β,γ) one of $\left(\frac{3}{2},\frac{1}{3},\frac{1}{2}\right)$, $\left(\frac{3}{2},\frac{1}{3},\frac{9}{2}\right)$ or $\left(\frac{3}{2},3,\frac{1}{2}\right)$ for $I(q)$ given by (5.5) and (α,β,γ) either $\left(6,\frac{3}{2},\frac{3}{2}\right)$ or $\left(\frac{2}{3},\frac{3}{2},\frac{3}{2}\right)$ for $I(q)$ given by (5.7).

When $\alpha=\beta=\gamma=0$, we have $\tau=0$. The first order system (5.1) is the classical Darboux–Halphen system and $y=-2(\Omega_1+\Omega_2+\Omega_3)$ satisfies Chazy's equation (2.1). This system arises as the anti-self-dual Ricci-flat equations for Bianchi-IX metrics (see [7, 22]).

6 Examples of flat $(2,3,5)$-distributions

In [19], Nurowski associated to $(2,3,5)$-distributions a conformal class of metrics of signature $(2,3)$. The fundamental curvature invariant of $(2,3,5)$-distributions appears as the Weyl tensor of Nurowski's metric. For $(2,3,5)$-distributions $\mathcal{D}_{F(q)}$ determined by a single function $F(q)$, the metric is described in [6, 19]. The distribution $\mathcal{D}_{F(q)}$ on M_{xyzpq} is encoded by the annihilator of the three 1-forms

$$\omega_1=\mathrm{d}y-p\mathrm{d}x,\qquad \omega_2=\mathrm{d}p-q\mathrm{d}x,\qquad \omega_3=\mathrm{d}z-F(q)\mathrm{d}x, \tag{6.1}$$

and supplemented by the 1-forms

$$\omega_4=\mathrm{d}q,\qquad \omega_5=\mathrm{d}x.$$

The coframe on M_{xyzpq} is given by

$$\begin{aligned}
&\theta^1=\omega_1-\frac{1}{F''}\left(F'\omega_2-\omega_3\right),\qquad \theta^2=\frac{1}{F''}\left(F'\omega_2-\omega_3\right),\\
&\theta^3=\left(1-\frac{F'F^{(3)}}{4(F'')^2}\right)\omega_2+\frac{F^{(3)}}{4(F'')^2}\omega_3,\\
&\theta^4=\left(\frac{7(F^{(3)})^2-4F''F^{(4)}}{40(F'')^3}\right)\left(F'\omega_2-\omega_3\right)+\omega_4-\omega_5,\qquad \theta^5=-\omega_4,
\end{aligned} \tag{6.2}$$

(cf. [6]) and Nurowski's metric

$$g_{\mathcal{D}_{F(q)}}=2\theta^1\theta^5-2\theta^2\theta^4+\frac{4}{3}\theta^3\theta^3 \tag{6.3}$$

has vanishing Weyl tensor (and hence conformally flat) iff $\mathcal{D}_{F(q)}$ has the split real form of G_2 as its group of local symmetries. There is a more elegant way to present Nurowski's metric for

the distribution $\mathcal{D}_{F(q)}$. Equivalently, we can encode the distribution by

$$\begin{aligned}
&\tilde{\omega}_1 = \omega_1 = \mathrm{d}y - p\mathrm{d}x, \\
&\tilde{\omega}_2 = \frac{1}{F''}(F'\omega_2 - \omega_3) = \frac{1}{F''}\left(F'(\mathrm{d}p - q\mathrm{d}x) - (\mathrm{d}z - F(q)\mathrm{d}x)\right), \\
&\tilde{\omega}_3 = \omega_2 = \mathrm{d}p - q\mathrm{d}x,
\end{aligned} \tag{6.4}$$

with $\tilde{\omega}_4 = \omega_4$ and $\tilde{\omega}_5 = \omega_5$. The coframe is then given by

$$\begin{aligned}
&\theta^1 = \tilde{\omega}_1 - \tilde{\omega}_2, \qquad \theta^2 = \tilde{\omega}_2, \qquad \theta^3 = \tilde{\omega}_3 - \frac{F^{(3)}}{4F''}\tilde{\omega}_2, \\
&\theta^4 = \left(\frac{7(F^{(3)})^2 - 4F''F^{(4)}}{40(F'')^2}\right)\tilde{\omega}_2 + \tilde{\omega}_4 - \tilde{\omega}_5, \qquad \theta^5 = -\tilde{\omega}_4.
\end{aligned}$$

Let us take

$$F(q) = \iint e^{\frac{1}{2}\int I(q)\mathrm{d}q}\mathrm{d}q\mathrm{d}q,$$

as in (4.6). This gives $F'' = e^{\frac{1}{2}\int I(q)\mathrm{d}q}$. Nurowski's metric now has a very simple form given by

$$\begin{aligned}
g_{\mathcal{D}_{F(q)}} &= 2\theta^1\theta^5 - 2\theta^2\theta^4 + \frac{4}{3}\theta^3\theta^3 \\
&= 2\tilde{\omega}_2\tilde{\omega}_5 - 2\tilde{\omega}_1\tilde{\omega}_4 + \frac{4}{3}(\tilde{\omega}_3)^2 - \frac{I}{3}\tilde{\omega}_2\tilde{\omega}_3 + \frac{1}{10}\left(I' - \frac{I^2}{6}\right)(\tilde{\omega}_2)^2.
\end{aligned} \tag{6.5}$$

The Ricci tensor of the above metric is

$$R_{ab}\theta^a\theta^b = \frac{9}{120}\left(6I' - I^2\right)(\tilde{\omega}_4)^2.$$

We can consider conformal rescalings of the metric such that $\hat{g}_{\mathcal{D}_{F(q)}} = \Omega^2 g_{\mathcal{D}_{F(q)}}$ is Ricci flat. It turns out that if we take $\Omega = \nu(q)^{-1} > 0$, then the Ricci tensor of the rescaled metric $\hat{g}_{\mathcal{D}_{F(q)}}$ is given by

$$R_{ab}\theta^a\theta^b = \frac{3}{40\nu}\left(40\nu'' + \left(6I' - I^2\right)\nu\right)(\tilde{\omega}_4)^2, \tag{6.6}$$

so the appropriate conformal scale $\nu(q)$ can be found by solving the differential equation in (6.6) (cf. [24, Proposition 35]). In the first part of this section we consider the conformally flat metrics (6.5) obtained by solving (1.2) using the solution (2.3). Next, we then consider the solutions obtained from different parametrisations of the generalised Chazy equation given by (5.4), (5.5) and (5.7). We also consider conformally flat metrics obtained from solving the Legendre transform of (1.2). This involves computing the coframe for the metric under the Legendre transform. Finally, we consider the metrics obtain from Chazy's solutions given by (3.2). The metrics associated to $(2,3,5)$-distributions $\mathcal{D}_{F(q)}$ of the form $F(q) = q^m$ where $m \in \left\{-1, \frac{1}{3}, \frac{2}{3}, 2\right\}$ are given in [18].

6.1 Chazy's solution

In order to express Nurowski's metric associated to flat $(2,3,5)$-distributions obtained from solving (1.2), we have to switch independent variable s and dependent variable q. In other words we pass to coordinates (x, y, z, p, s) with $q(s) = \frac{z_2(s)}{z_1(s)}$ where $z_1(s)$, $z_2(s)$ are given in Corollary 3.2.

Let us first consider solutions of the form

$$I(q(s)) = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1(s)$$

as in Corollary 3.2. Observe that

$$\int I(q)\mathrm{d}q = 6\log z_1$$

and so

$$F(q) = \iint \mathrm{e}^{\frac{1}{2}\int I(q)\mathrm{d}q}\mathrm{d}q\mathrm{d}q = \iint (z_1)^3\mathrm{d}q\mathrm{d}q.$$

For this parametrisation we have

$$\mathrm{d}q = \frac{z_1\dot{z}_2 - z_2\dot{z}_1}{(z_1)^2}\mathrm{d}s,$$

so that

$$F(q(s)) = \int\left(\int z_1(z_1\dot{z}_2 - z_2\dot{z}_1)\mathrm{d}s\right)\frac{z_1\dot{z}_2 - z_2\dot{z}_1}{(z_1)^2}\mathrm{d}s.$$

Let us denote

$$K(s) = \int z_1(z_1\dot{z}_2 - z_2\dot{z}_1)\mathrm{d}s.$$

Theorem 6.1. *Let $z_1(s)$, $z_2(s)$ be two linearly independent solutions to (2.2) with (a,b,c) given by one of the list in Corollary 3.2. Let $\mathcal{D}_C$ denote the $(2,3,5)$-distribution on M_{xyzps} associated to the annihilator of*

$$\begin{aligned}
\tilde{\omega}_1 &= \mathrm{d}y - p\mathrm{d}x,\\
\tilde{\omega}_2 &= \frac{K(s)}{(z_1)^3}\left(\mathrm{d}p - \frac{z_2}{z_1}\mathrm{d}x\right) - \frac{1}{(z_1)^3}\left(\mathrm{d}z - \left(\int K(s)\frac{z_1\dot{z}_2 - z_2\dot{z}_1}{(z_1)^2}\mathrm{d}s\right)\mathrm{d}x\right),\\
\tilde{\omega}_3 &= \mathrm{d}p - \frac{z_2}{z_1}\mathrm{d}x.
\end{aligned}$$

Supplement by the 1-forms

$$\tilde{\omega}_4 = \frac{z_1\dot{z}_2 - z_2\dot{z}_1}{(z_1)^2}\mathrm{d}s, \qquad \tilde{\omega}_5 = \mathrm{d}x.$$

Then Nurowski's metric (6.5)

$$\begin{aligned}
g_{\mathcal{D}_C} = {}& 2\tilde{\omega}_2\tilde{\omega}_5 - 2\tilde{\omega}_1\tilde{\omega}_4 + \frac{4}{3}(\tilde{\omega}_3)^2 - \frac{2z_1\dot{z}_1}{z_1\dot{z}_2 - z_2\dot{z}_1}\tilde{\omega}_2\tilde{\omega}_3\\
&+ \frac{3}{5}\left(\frac{(z_1)^3\ddot{z}_1}{(z_1\dot{z}_2 - z_2\dot{z}_1)^2} - \frac{(z_1)^3\dot{z}_2(z_1\ddot{z}_2 - z_2\ddot{z}_1)}{(z_1\dot{z}_2 - z_2\dot{z}_1)^3}\right)(\tilde{\omega}_2)^2
\end{aligned}$$

has vanishing Weyl tensor (and hence conformally flat) and $\mathcal{D}_C$ has the split real form of G_2 as its group of local symmetries.

Let us provide an explicit example given by Corollary 3.2. It turns out for the values of (a,b,c) obtained in Corollary 3.2, the solutions can be given by elementary functions. For $(a,b,c)=\left(-\frac{2}{3},\frac{5}{6},\frac{1}{2}\right)$, the solutions to the hypergeometric differential equation (2.2) are given by

$$z(s)=\mu(s-1)^{\frac{1}{6}}P_1^{\frac{1}{3}}(\sqrt{s})+\nu(s-1)^{\frac{1}{6}}Q_1^{\frac{1}{3}}(\sqrt{s}),$$

where P_ℓ^m and Q_ℓ^m are the associated Legendre functions. This suggest passing further to the variable $r=\sqrt{s}$, in which case (2.2) with $(a,b,c)=\left(-\frac{2}{3},\frac{5}{6},\frac{1}{2}\right)$ becomes

$$\frac{1}{4}(1-r^2)z''(r)-\frac{1}{3}rz'(r)+\frac{5}{9}z(r)=0. \tag{6.7}$$

The general solution is now given by the elementary functions

$$z(r)=c_1(r-1)^{\frac{1}{3}}(3r+1)+c_2(r+1)^{\frac{1}{3}}(3r-1).$$

There is thus a 4-dimensional space of solutions given by

$$z_1(r)=c_1(r-1)^{\frac{1}{3}}(3r+1)+c_2(r+1)^{\frac{1}{3}}(3r-1),$$
$$z_2(r)=c_3(r-1)^{\frac{1}{3}}(3r+1)+c_4(r+1)^{\frac{1}{3}}(3r-1),$$

where $c_1c_4-c_2c_3\neq 0$. We also note that in the case $(a,b,c)=\left(-\frac{2}{3},\frac{5}{6},\frac{2}{3}\right)$, the change of variable $r=\sqrt{1-s}$ brings (2.2) to (6.7), so that the two hypergeometric ODEs with $(a,b,c)=\left(-\frac{2}{3},\frac{5}{6},\frac{1}{2}\right)$ and $\left(-\frac{2}{3},\frac{5}{6},\frac{2}{3}\right)$ can be brought to the same equation (6.7) by a coordinate transformation.

Moreover, since

$${}_2F_1\left(-\frac{2}{3},\frac{5}{6};\frac{2}{3};4s(1-s)\right)={}_2F_1\left(-\frac{4}{3},\frac{5}{3};\frac{2}{3};s\right),$$

if we take $r=\sqrt{1-4s(1-s)}=2s-1$, we can pass from (2.2) with $(a,b,c)=\left(-\frac{4}{3},\frac{5}{3};\frac{2}{3}\right)$ to equation (6.7). Also compare with the second row of Table 1.

We now pass to coordinates on M_{xyzpr} and take for a simple example

$$z_1(r)=c_1(r-1)^{\frac{1}{3}}(3r+1),\qquad z_2(r)=c_2(r+1)^{\frac{1}{3}}(3r-1)$$

where c_1, c_2 are non-zero constants. We also make use of

$$\mathrm{d}s=2r\mathrm{d}r.$$

Here we have $K(r)=-16(c_1)^2c_2(r-1)^{\frac{2}{3}}(r+1)^{\frac{1}{3}}$. We have

Proposition 6.2. *Let c_1, c_2 be non-zero constants. Let $\mathcal{D}_0$ denote the $(2,3,5)$-distribution on M_{xyzpr} associated to the annihilator of*

$$\tilde\omega_1=\mathrm{d}y-p\mathrm{d}x,$$
$$\tilde\omega_2=\frac{-16c_2(r+1)^{\frac{1}{3}}}{c_1(r-1)^{\frac{1}{3}}(3r+1)^3}\mathrm{d}p-\frac{1}{(c_1)^3(r-1)(3r+1)^3}\mathrm{d}z+\frac{16(c_2)^2(r+1)^{\frac{2}{3}}}{(c_1)^2(r-1)^{\frac{2}{3}}(3r+1)^3}\mathrm{d}x,$$
$$\tilde\omega_3=\mathrm{d}p-\frac{c_2(r+1)^{\frac{1}{3}}(3r-1)}{c_1(r-1)^{\frac{1}{3}}(3r+1)}\mathrm{d}x,$$

and supplemented by the 1-forms

$$\tilde\omega_4=-\frac{16c_2}{3c_1(3r+1)^2(r-1)^{\frac{4}{3}}(r+1)^{\frac{2}{3}}}\mathrm{d}r,\qquad \tilde\omega_5=\mathrm{d}x.$$

Then Nurowski's metric (6.5)

$$g_{\mathcal{D}_0} = 2\tilde{\omega}_2\tilde{\omega}_5 - 2\tilde{\omega}_1\tilde{\omega}_4 + \frac{4}{3}(\tilde{\omega}_3)^2 + \frac{c_1(3r-2)(3r+1)(r-1)^{\frac{1}{3}}(r+1)^{\frac{2}{3}}}{2c_2}\tilde{\omega}_2\tilde{\omega}_3 + \frac{3(c_1)^2(r-1)^{\frac{5}{3}}(3r+1)^4(r+1)^{\frac{1}{3}}}{64(c_2)^2}(\tilde{\omega}_2)^2$$

has vanishing Weyl tensor (and hence conformally flat) and $\mathcal{D}_0$ has the split real form of G_2 as its group of local symmetries. The Ricci tensor for this metric is

$$R_{ab}\theta^a\theta^b = \frac{6}{r^2-1}\mathrm{d}r\mathrm{d}r.$$

Rescaling this metric by

$$\Omega = \frac{1}{\nu} = \frac{4}{3}\frac{(3r+1)(r-1)^{\frac{1}{3}}}{a_1(r-1)^{\frac{1}{3}} - a_2(r+1)^{\frac{1}{3}}},$$

where a_1 and a_2 are constants, the conformally rescaled metric $\hat{g}_{\mathcal{D}_0} = \Omega^2 g_{\mathcal{D}_0}$ given by

$$\hat{g}_{\mathcal{D}_0} = \frac{16(3r+1)^2(r-1)^{\frac{2}{3}}}{9(a_1(r-1)^{\frac{1}{3}} - a_2(r+1)^{\frac{1}{3}})^2}\Bigg(2\tilde{\omega}_2\tilde{\omega}_5 - 2\tilde{\omega}_1\tilde{\omega}_4 + \frac{4}{3}(\tilde{\omega}_3)^2 + \frac{c_1(3r-2)(3r+1)(r-1)^{\frac{1}{3}}(r+1)^{\frac{2}{3}}}{2c_2}\tilde{\omega}_2\tilde{\omega}_3 + \frac{3(c_1)^2(r-1)^{\frac{5}{3}}(3r+1)^4(r+1)^{\frac{1}{3}}}{64(c_2)^2}(\tilde{\omega}_2)^2\Bigg)$$

is both Ricci-flat and conformally flat.

6.2 Other parametrisations of the generalised Chazy equation

Instead of choosing $I(q) = 6\frac{\mathrm{d}}{\mathrm{d}q}\log z_1$, we can consider the parametrisation

$$I(q) = \frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}^3}{s^2(s-1)^2}$$

given in (5.4). In this case

$$F(q) = \iint \mathrm{e}^{\frac{1}{2}\int I(q)\mathrm{d}q}\mathrm{d}q\mathrm{d}q = \iint \frac{\dot{s}^{\frac{3}{2}}}{s(s-1)}\mathrm{d}q\mathrm{d}q$$

and the corresponding metric associated to the $(2,3,5)$-distribution $\mathcal{D}_{F(q)}$ gives the following

Theorem 6.3. *Let $s(q)$ be a solution to* (5.2) *with (α,β,γ) given by one of*

$$(3,3,3), \qquad \left(3,\frac{1}{3},\frac{1}{3}\right), \qquad \left(\frac{1}{3},3,\frac{1}{3}\right), \qquad \left(\frac{1}{3},\frac{1}{3},3\right).$$

Let $\mathcal{D}_s$ denote the $(2,3,5)$-distribution on M_{xyzpq} associated to the annihilator of

$$\omega_1 = \mathrm{d}y - p\mathrm{d}x, \qquad \omega_2 = \mathrm{d}p - q\mathrm{d}x, \qquad \omega_3 = \mathrm{d}z - \left(\iint \frac{\dot{s}^{\frac{3}{2}}}{s(s-1)}\mathrm{d}q\mathrm{d}q\right)\mathrm{d}x.$$

Supplement by the 1-forms

$$\omega_4 = \mathrm{d}q, \qquad \omega_5 = \mathrm{d}x,$$

and take the coframe on M_{xyzpq} to be given by $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ as in (6.2) where

$$F(q)=\iint \frac{\dot{s}^{\frac{3}{2}}}{s(s-1)}\mathrm{d}q\mathrm{d}q.$$

Then Nurowski's metric

$$g_{\mathcal{D}_s}=2\theta^1\theta^5-2\theta^2\theta^4+\frac{4}{3}\theta^3\theta^3$$

has vanishing Weyl tensor (and hence conformally flat) and $\mathcal{D}_s$ has the split real form of G_2 as its group of local symmetries.

To obtain explicit examples, it is useful to switch the independent variable q and the dependent variable s. We pass to the variables (x,y,z,p,s) with $q=\frac{u_2(s)}{u_1(s)}$ where u_1 and u_2 are linearly independent solutions of (5.3) given in Table 1. Note that up to fractional linear transformations in the variable s, we only need to consider the solutions to (5.2) with the values of (α,β,γ) given by either $(3,3,3)$ or $\left(3,\frac{1}{3},\frac{1}{3}\right)$ for the parametrisation given by (5.4). Note the symmetry permuting β and γ.

A computation shows that $W(u_1,u_2)=u_1\dot{u}_2-\dot{u}_1u_2$ is constant, which we can normalise to set $W(u_1,u_2)=1$. We have

$$\mathrm{d}q=\frac{u_1\dot{u}_2-u_2\dot{u}_1}{(u_1)^2}\mathrm{d}s=\frac{1}{(u_1)^2}\mathrm{d}s$$

and

$$\dot{s}=\frac{1}{q'(s)}=(u_1)^2,$$

so that

$$F(q(s))=\int\left(\int\frac{(u_1)^3}{s(s-1)}\frac{1}{(u_1)^2}\mathrm{d}s\right)\frac{1}{(u_1)^2}\mathrm{d}s=\int\left(\int\frac{u_1}{s(s-1)}\mathrm{d}s\right)\frac{1}{(u_1)^2}\mathrm{d}s.$$

Let us denote

$$K(s)=\int\frac{u_1}{s(s-1)}\mathrm{d}s.$$

Theorem 6.4. *Let $u_1(s)$, $u_2(s)$ be two linearly independent solutions to (5.3) subject to the constraint $W(u_1,u_2)=1$ with (α,β,γ) given by Table 1. Let $\mathcal{D}_s$ denote the $(2,3,5)$-distribution on M_{xyzps} associated to the annihilator of*

$$\omega_1=\mathrm{d}y-p\mathrm{d}x,\qquad \omega_2=\mathrm{d}p-\frac{u_2(s)}{u_1(s)}\mathrm{d}x,\qquad \omega_3=\mathrm{d}z-\left(\int\frac{K(s)}{(u_1)^2}\mathrm{d}s\right)\mathrm{d}x.$$

We pass to the annihilator 1-forms given by (6.4) to obtain

$$\tilde{\omega}_1=\omega_1=\mathrm{d}y-p\mathrm{d}x,\qquad \tilde{\omega}_2=\frac{s(s-1)}{(u_1)^3}(K\omega_2-\omega_3),\qquad \tilde{\omega}_3=\omega_2=\mathrm{d}p-\frac{u_2(s)}{u_1(s)}\mathrm{d}x,$$

with

$$\tilde{\omega}_4=\omega_4=\frac{1}{(u_1)^2}\mathrm{d}s,\qquad \tilde{\omega}_5=\omega_5=\mathrm{d}x.$$

Take the coframe on M_{xyzps} to be given by

$$\theta^1 = \tilde{\omega}_1 - \tilde{\omega}_2, \qquad \theta^2 = \tilde{\omega}_2, \qquad \theta^3 = \tilde{\omega}_3 - \frac{(u_1)^2}{4s(s-1)}\left(3\frac{\dot{u}_1}{u_1}s(s-1) - (2s-1)\right)\tilde{\omega}_2,$$
$$\theta^4 = \frac{(u_1)^4}{40}\left(\frac{4s^2-4s-1}{s^2(s-1)^2} + 3V(s) - \frac{10(2s-1)}{s(s-1)}\frac{\dot{u}_1}{u_1} + 15\left(\frac{\dot{u}_1}{u_1}\right)^2\right)\tilde{\omega}_2 + \tilde{\omega}_4 - \tilde{\omega}_5,$$
$$\theta^5 = -\tilde{\omega}_4.$$

Then Nurowski's metric

$$\begin{aligned} g_{\mathcal{D}_s} &= 2\theta^1\theta^5 - 2\theta^2\theta^4 + \frac{4}{3}\theta^3\theta^3 \\ &= 2\tilde{\omega}_2\tilde{\omega}_5 - 2\tilde{\omega}_1\tilde{\omega}_4 + \frac{4}{3}(\tilde{\omega}_3)^2 + \frac{2u_1(3s(1-s)u_1' + (2s-1)u_1)}{3s(s-1)}\tilde{\omega}_2\tilde{\omega}_3 \\ &\quad - \frac{(u_1)^4(9V(s)s^2(s-1)^2 - 8(s^2-s+1))}{60(s-1)^2s^2}(\tilde{\omega}_2)^2 \end{aligned}$$

has vanishing Weyl tensor (and hence conformally flat) and $\mathcal{D}_s$ has the split real form of G_2 as its group of local symmetries.

The analogous results hold for $I(q)$ given by the formulas in (5.5), (5.6) and (5.7). If we take

$$I(q) = \frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}^3}{s^2(s-1)^{\frac{3}{2}}},$$

as in (5.5), we have the corresponding $(2,3,5)$-distribution associated to

$$F(q) = \iint \frac{\dot{s}^{\frac{3}{2}}}{s(s-1)^{\frac{3}{4}}}\mathrm{d}q\mathrm{d}q.$$

Theorem 6.5. *Let $s(q)$ be a solution to (5.2) with (α,β,γ) given by $\left(\frac{3}{2},\frac{1}{3},\frac{1}{2}\right)$, $\left(\frac{3}{2},3,\frac{1}{2}\right)$ or $\left(\frac{3}{2},\frac{1}{3},\frac{9}{2}\right)$. Let $\mathcal{D}_{s_1}$ denote the $(2,3,5)$-distribution on M_{xyzpq} associated to the annihilator of*

$$\omega_1 = \mathrm{d}y - p\mathrm{d}x, \qquad \omega_2 = \mathrm{d}p - q\mathrm{d}x, \qquad \omega_3 = \mathrm{d}z - \left(\iint \frac{\dot{s}^{\frac{3}{2}}}{s(s-1)^{\frac{3}{4}}}\mathrm{d}q\mathrm{d}q\right)\mathrm{d}x.$$

Supplement by the 1-forms

$$\omega_4 = \mathrm{d}q, \qquad \omega_5 = \mathrm{d}x,$$

and take the coframe on M_{xyzpq} to be given by $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ as in (6.2). Then Nurowski's metric (6.3) has vanishing Weyl tensor (and hence conformally flat) and $\mathcal{D}_{s_1}$ has the split real form of G_2 as its group of local symmetries.

Similarly, for $I(q)$ given by (5.7), that is to say

$$I(q) = \frac{\mathrm{d}}{\mathrm{d}q}\log\frac{\dot{s}^3}{s^{\frac{5}{2}}(s-1)^{\frac{5}{2}}},$$

we have the corresponding $(2,3,5)$-distribution associated to

$$F(q) = \iint \frac{\dot{s}^{\frac{3}{2}}}{s^{\frac{5}{4}}(s-1)^{\frac{5}{4}}}\mathrm{d}q\mathrm{d}q.$$

Theorem 6.6. *Let $s(q)$ be a solution to (5.2) with (α,β,γ) given by $\left(6,\frac{3}{2},\frac{3}{2}\right)$, $\left(\frac{2}{3},\frac{3}{2},\frac{3}{2}\right)$ or $\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right)$. Let $\mathcal{D}_{s_2}$ denote the $(2,3,5)$-distribution on M_{xyzpq} associated to the annihilator of*

$$\omega_1 = \mathrm{d}y - p\mathrm{d}x, \qquad \omega_2 = \mathrm{d}p - q\mathrm{d}x, \qquad \omega_3 = \mathrm{d}z - \left(\iint \frac{\dot{s}^{\frac{3}{2}}}{s^{\frac{5}{4}}(s-1)^{\frac{5}{4}}}\mathrm{d}q\mathrm{d}q\right)\mathrm{d}x.$$

Supplement by the 1-forms

$$\omega_4 = \mathrm{d}q, \qquad \omega_5 = \mathrm{d}x,$$

and take the coframe on M_{xyzpq} to be given by $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ as in (6.2). Then Nurowski's metric (6.3) has vanishing Weyl tensor (and hence conformally flat) and $\mathcal{D}_{s_2}$ has the split real form of G_2 as its group of local symmetries.

6.3 Legendre transformed coframe

The Legendre transform of Proposition 4.1 takes the 1-forms (6.1) to

$$\omega_1 = \mathrm{d}y - p\mathrm{d}x, \qquad \omega_2 = \mathrm{d}p - H'\mathrm{d}x, \qquad \omega_3 = \mathrm{d}z - (tH' - H)\mathrm{d}x,$$
$$\omega_4 = H''\mathrm{d}t, \qquad \omega_5 = \mathrm{d}x,$$

where $H = H(t)$ with $H'' \neq 0$ on M_{xyzpt} and the coframe (6.2) to

$$\theta^1 = \omega_1 - H''(t\omega_2 - \omega_3), \qquad \theta^2 = H''(t\omega_2 - \omega_3), \qquad \theta^3 = \left(1 + t\frac{H'''}{4H''}\right)\omega_2 - \frac{H'''}{4H''}\omega_3,$$
$$\theta^4 = \frac{4H''H'''' - 5(H''')^2}{40(H'')^3}(t\omega_2 - \omega_3) + \omega_4 - \omega_5, \qquad \theta^5 = -\omega_4.$$

Note that our $H(t)$ is related to $\Theta(x_5)$ of [6] via $\Theta_{55} = -H$, $t = x_5$. The Nurowski metric

$$g = 2\theta^1\theta^5 - 2\theta^2\theta^4 + \frac{4}{3}\theta^3\theta^3$$

has the only non-vanishing component of the Weyl tensor given by the left hand side term of the dual ODE (4.2). This accounts for the appearance of equation (4.3) in [9]. The solutions of the dual generalised Chazy ODE (4.5) with parameter $\pm\frac{3}{2}$ give us further examples of flat $(2,3,5)$-distributions. We pass to (x,y,z,p,s) as before, with $t(s) = \frac{w_2(s)}{w_1(s)}$ where $w_1(s)$, $w_2(s)$ are linearly independent solutions to (2.2) with (a,b,c) one of

$$\left(-\frac{1}{4},\frac{5}{12},\frac{1}{2}\right), \qquad \left(-\frac{1}{4},\frac{5}{12},\frac{2}{3}\right), \qquad \left(-\frac{1}{2},\frac{5}{6},\frac{2}{3}\right).$$

Here we have taken $k = \frac{3}{2}$. Note that the equations (2.2) for $(a,b,c) = \left(-\frac{1}{4},\frac{5}{12},\frac{1}{2}\right)$ and $\left(-\frac{1}{4},\frac{5}{12},\frac{2}{3}\right)$ are equivalent by a linear transformation and thus the solutions to each equation can be expressed as linear combinations of the other, while the solutions for $(a,b,c) = \left(-\frac{1}{4},\frac{5}{12},\frac{2}{3}\right)$ and $\left(-\frac{1}{2},\frac{5}{6},\frac{2}{3}\right)$ are equivalent by a quadratic transformation as before. However, the author does not know if the solutions in these cases can be expressed by elementary functions. We consider once again solutions to (4.5) of the form $u(t) = 6\frac{\mathrm{d}}{\mathrm{d}t}\log w_1$. This gives

$$H(t) = \iint (w_1)^4\mathrm{d}t\mathrm{d}t, \qquad H'(t) = \int (w_1)^4\mathrm{d}t, \qquad H''(t) = (w_1)^4.$$

For this parametrisation we have

$$\mathrm{d}t = \frac{w_1\dot{w}_2 - w_2\dot{w}_1}{(w_1)^2}\mathrm{d}s$$

and so

$$H = \iint (w_1)^2(w_1\dot{w}_2 - w_2\dot{w}_1)\mathrm{d}s\frac{w_1\dot{w}_2 - w_2\dot{w}_1}{(w_1)^2}\mathrm{d}s, \qquad H' = \int (w_1)^2(w_1\dot{w}_2 - w_2\dot{w}_1)\mathrm{d}s.$$

We therefore obtain

$$\begin{aligned}
&\omega_1 = \mathrm{d}y - p\mathrm{d}x, \qquad \omega_2 = \mathrm{d}p - \int (w_1)^2(w_1\dot{w}_2 - w_2\dot{w}_1)\mathrm{d}s\mathrm{d}x,\\
&\omega_3 = \mathrm{d}z - \left(\frac{w_2}{w_1}\int (w_1)^2(w_1\dot{w}_2 - w_2\dot{w}_1)\mathrm{d}s\right.\\
&\qquad \left. - \iint (w_1)^2(w_1\dot{w}_2 - w_2\dot{w}_1)\mathrm{d}s\frac{w_1\dot{w}_2 - w_2\dot{w}_1}{(w_1)^2}\mathrm{d}s\right)\mathrm{d}x,\\
&\omega_4 = (w_1)^2(w_1\dot{w}_2 - w_2\dot{w}_1)\mathrm{d}s, \qquad \omega_5 = \mathrm{d}x
\end{aligned}$$

and the adapted coframe for Nurowski's metric is

$$\begin{aligned}
&\theta^1 = \omega_1 - (w_1)^4\left(\frac{w_2}{w_1}\omega_2 - \omega_3\right), \qquad \theta^2 = (w_1)^4\left(\frac{w_2}{w_1}\omega_2 - \omega_3\right),\\
&\theta^3 = \left(1 + \frac{w_2\dot{w}_1}{w_1\dot{w}_2 - w_2\dot{w}_1}\right)\omega_2 - \frac{w_1\dot{w}_1}{w_1\dot{w}_2 - w_2\dot{w}_1}\omega_3,\\
&\theta^4 = \frac{2(\ddot{w}_1\dot{w}_2 - \ddot{w}_2\dot{w}_1)}{5(w_1\dot{w}_2 - w_2\dot{w}_1)^3}\left(\frac{w_2}{w_1}\omega_2 - \omega_3\right) + \omega_4 - \omega_5, \qquad \theta^5 = -\omega_4.
\end{aligned} \tag{6.8}$$

Equivalently, we can take

$$\tilde{\omega}_1 = \omega_1, \qquad \tilde{\omega}_2 = (w_1)^4\left(\frac{w_2}{w_1}\omega_2 - \omega_3\right), \qquad \tilde{\omega}_3 = \omega_2$$

with $\tilde{\omega}_4 = \omega_4$ and $\tilde{\omega}_5 = \omega_5$. The coframe is then given by

$$\begin{aligned}
&\theta^1 = \tilde{\omega}_1 - \tilde{\omega}_2, \qquad \theta^2 = \tilde{\omega}_2, \qquad \theta^3 = \tilde{\omega}_3 + \frac{\dot{w}_1}{(w_1)^3(w_1\dot{w}_2 - w_2\dot{w}_1)}\tilde{\omega}_2,\\
&\theta^4 = \frac{2(\ddot{w}_1\dot{w}_2 - \ddot{w}_2\dot{w}_1)}{5(w_1)^4(w_1\dot{w}_2 - w_2\dot{w}_1)^3}\tilde{\omega}_2 + \tilde{\omega}_4 - \tilde{\omega}_5, \qquad \theta^5 = -\tilde{\omega}_4.
\end{aligned} \tag{6.9}$$

Proposition 6.7. *The Nurowski metric*

$$g = 2\theta^1\theta^5 - 2\theta^2\theta^4 + \frac{4}{3}\theta^3\theta^3$$

given by the above coframe (6.8) for $w_1(s)$, $w_2(s)$ linearly independent solutions to the hypergeometric differential equation (2.2) with (a,b,c) one of

$$\left(-\frac{1}{4}, \frac{5}{12}, \frac{1}{2}\right), \qquad \left(-\frac{1}{4}, \frac{5}{12}, \frac{2}{3}\right), \qquad \left(-\frac{1}{2}, \frac{5}{6}, \frac{2}{3}\right)$$

are all conformally flat. For each (a,b,c) there is a 4-dimensional family of solutions. Up to fractional linear transformation in the variable s there are 2 distinct classes given by the last two entries.

In addition, the Legendre transformation of Lemma 4.2 given by

$$w_1 = (z_1)^{-\frac{3}{4}}, \qquad w_2 = (z_1)^{-\frac{3}{4}}\int z_1(z_1\dot{z}_2 - z_2\dot{z}_1)\mathrm{d}s$$

takes the coframe (6.9) to the coframe in Theorem 6.1 and conversely so. The Legendre transform also applies to the coframes given in Theorems 6.3, 6.5 and 6.6. We also have the analogous results of Section 6.2. Up to fractional linear transformations in s we have 7 classes of solutions to the generalised Chazy equation (4.5) determined by $s(t)$ satisfying (5.2). These are given by the parametrisations in Section 5 and the corresponding values for (α,β,γ) can be computed for the parameter $k=\frac{3}{2}$. For the parametrisation

$$H(t)=\iint \frac{\dot{s}^2}{s^{\frac{4}{3}}(s-1)^{\frac{4}{3}}}\mathrm{d}t\mathrm{d}t,$$

the values for (α,β,γ) are given by either $\left(\frac{4}{3},\frac{4}{3},\frac{4}{3}\right)$ or $\left(\frac{4}{3},\frac{1}{3},\frac{1}{3}\right)$. For

$$H(t)=\iint \frac{\dot{s}^2}{s^{\frac{4}{3}}(s-1)}\mathrm{d}t\mathrm{d}t,$$

(α,β,γ) takes the values of $\left(\frac{2}{3},\frac{1}{2},\frac{1}{3}\right)$, $\left(\frac{2}{3},\frac{1}{2},\frac{4}{3}\right)$ or $\left(\frac{2}{3},2,\frac{1}{3}\right)$. For

$$H(t)=\iint \frac{\dot{s}^2}{s^{\frac{5}{3}}(s-1)^{\frac{5}{3}}}\mathrm{d}t\mathrm{d}t,$$

we obtain $\left(\frac{8}{3},\frac{2}{3},\frac{2}{3}\right)$ or $\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$.

The Legendre transform therefore provides seven further classes of flat Nurowski metrics up to fractional linear transformations in s.

6.4 Additional examples

The solution (3.2) for $k=\pm\frac{2}{3}$ gives $I(q)=-\frac{8}{3(q+C)}-\frac{10}{3(q+B)}$. Hence, the metric (6.5) on M_{xyzpq} given by

$$\begin{aligned} g_{\mathcal{D}_{F(q)}} &= 2\tilde{\omega}_2\tilde{\omega}_5-2\tilde{\omega}_1\tilde{\omega}_4+\frac{4}{3}(\tilde{\omega}_3)^2\\ &\quad+\left(\frac{8}{9(q+C)}+\frac{10}{9(q+B)}\right)\tilde{\omega}_2\tilde{\omega}_3+\frac{4(B-C)^2}{27(q+B)^2(q+C)^2}(\tilde{\omega}_2)^2 \end{aligned}$$

is conformally flat. In the dual coframe the flat metric (6.5) on M_{xyzpt} is given by

$$\begin{aligned} g &= 2\tilde{\omega}_2\tilde{\omega}_5-2\tilde{\omega}_1\tilde{\omega}_4+\frac{4}{3}(\tilde{\omega}_3)^2\\ &\quad+\frac{4}{9}u(t)e^{\int-\frac{2}{3}u(t)\mathrm{d}t}\tilde{\omega}_2\tilde{\omega}_3-\frac{2}{135}(9\dot{u}(t)-4u(t)^2)e^{\int-\frac{4}{3}u(t)\mathrm{d}t}(\tilde{\omega}_2)^2, \end{aligned} \tag{6.10}$$

where $u(t)$ satisfies the generalised Chazy equation (4.5) with parameter $k=\pm\frac{3}{2}$. The solution (3.2) for $k=\pm\frac{3}{2}$ gives $u(t)=-\frac{15}{4(t-a)}-\frac{9}{4(t-b)}$ and substituting this into (6.10) gives the conformally flat metric

$$\begin{aligned} g &= 2\tilde{\omega}_2\tilde{\omega}_5-2\tilde{\omega}_1\tilde{\omega}_4+\frac{4}{3}(\tilde{\omega}_3)^2-\frac{256}{3}(8t-5b-3a)(t-a)^{\frac{3}{2}}(t-b)^{\frac{1}{2}}\tilde{\omega}_2\tilde{\omega}_3\\ &\quad+\frac{65536}{3}(t-b)^2(t-a)^3(4t-3a-b)(\tilde{\omega}_2)^2. \end{aligned}$$

To summarise the results of this section, we first presented different examples of Nurowski metrics that are conformally flat up to fractional linear transformation in the variable s. Two examples are given in Theorem 6.3, three examples are given in Theorem 6.5 and two more in Theorem 6.6. Seven additional examples are obtained from the Legendre transform as in Proposition 4.1. We also have 2 additional examples from the solutions of the form (3.2). Finally, there are examples associated to distributions of the form $F(q)=q^m$, where $m\in\left\{-1,\frac{1}{3},\frac{2}{3},2\right\}$ and passing to the dual coframe, distributions of the form $H(t)=t^m$, where $m\in\left\{-2,-\frac{1}{2},\frac{1}{2},2\right\}$.

7 An–Nurowski circle twistor bundle

In [5], An and Nurowski showed how to associate to a split signature conformal structure $[g]$ on a 4-manifold M^4 a natural $(2,3,5)$-distribution. 4-dimensional split signature conformal structures admit real self-dual totally null 2-planes. The bundle of such 2-planes is a circle bundle over M^4 with fibres S^1 [5]. This is called the circle twistor bundle $\mathbb{T}(M^4)$ and it has a rank 2 distribution given by lifting horizontally the null 2-planes on M^4. This distribution is non-integrable, i.e., defines a $(2,3,5)$-distribution whenever the self-dual part of the Weyl tensor of g on M^4 is non-vanishing. Moreover in [6], the authors presented split signature conformal structures on M^4 that give rise to $(2,3,5)$-distributions of the form $\mathcal{D}_{F(q)}$ on $\mathbb{T}(M^4)$. Such split signature metrics are called Plebański's second heavenly metrics in [6].

Following [6, Section 3], we can find these metrics that have a flat circle twistor bundle. Such circle twistor bundles have split G_2 as their group of symmetries. Let (w,x,y,z) be local coordinates on M^4. Let $\Theta=\Theta(w,x,y,z)$ be an arbitrary function of 4 variables (second heavenly function of Plebański). Let (e_i) be an orthonormal frame on M^4 and (θ^j) the dual coframe satisfying $\theta^j(e_i)=\delta^j{}_i$. The split signature Plebański metric is given by

$$g=g_{ij}\theta^i\otimes\theta^j=2\theta^1\theta^2+2\theta^3\theta^4,$$

where $\theta^i\theta^j=\frac{1}{2}\theta^i\otimes\theta^j+\frac{1}{2}\theta^j\otimes\theta^i$ and

$$\theta^1=\mathrm{d}x-\Theta_{yy}\mathrm{d}w+\Theta_{xy}\mathrm{d}z,\qquad \theta^2=\mathrm{d}w,\qquad \theta^3=\mathrm{d}y-\Theta_{xx}\mathrm{d}z+\Theta_{xy}\mathrm{d}w,\qquad \theta^4=\mathrm{d}z.$$

Hence $g_{12}=g_{34}=1$ and all other components are zero. Such split signature metrics admit a real parallel spinor [15]. A computation shows that the connection 1-forms we need are given by

$$\Gamma^1{}_1=-\Theta_{yyx}\theta^2+\Theta_{yxx}\theta^4,\qquad \Gamma^1{}_3=-\Theta_{yyy}\theta^2+\Theta_{yyx}\theta^4,$$
$$\Gamma^3{}_1=\Theta_{yxx}\theta^2-\Theta_{xxx}\theta^4,\qquad \Gamma^3{}_3=\Theta_{yyx}\theta^2-\Theta_{yxx}\theta^4.$$

Using [6] and Nurowski's notes [20], we find that the $(2,3,5)$-distribution on $\mathbb{T}(M^4)$ is annihilated by the following three 1-forms:

$$\begin{aligned}\omega^3&=\mathrm{d}\xi+\Gamma^3{}_1+\big(\Gamma^3{}_3-\Gamma^1{}_1\big)\xi-\Gamma^1{}_3\xi^2\\&=\mathrm{d}\xi+\big(\Theta_{yxx}+2\Theta_{yyx}\xi+\Theta_{yyy}\xi^2\big)\theta^2-\big(\Theta_{xxx}+2\Theta_{yxx}\xi+H_{yyx}\xi^2\big)\theta^4\\&=\mathrm{d}\xi+\big(\Theta_{yxx}+2\Theta_{yyx}\xi+\Theta_{yyy}\xi^2\big)\mathrm{d}w-\big(\Theta_{xxx}+2\Theta_{yxx}\xi+\Theta_{yyx}\xi^2\big)\mathrm{d}z\end{aligned}$$

and

$$\begin{aligned}\omega^4&=\xi\theta^4+\theta^2=\xi\mathrm{d}z+\mathrm{d}w,\\ \omega^5&=\theta^3-\xi\theta^1=\mathrm{d}y-\Theta_{xx}\mathrm{d}z+\Theta_{xy}\mathrm{d}w-\xi(\mathrm{d}x-\Theta_{yy}\mathrm{d}w+\Theta_{xy}\mathrm{d}z)\\&=\mathrm{d}y-\xi\mathrm{d}x-(\Theta_{xx}+\xi\Theta_{xy})\mathrm{d}z+(\Theta_{xy}+\xi\Theta_{yy})\mathrm{d}w.\end{aligned}$$

The distribution is therefore annihilated by the 1-forms

$$\tilde\omega^3=\mathrm{d}\xi-\big(\Theta_{xxx}+3\Theta_{yxx}\xi+3\Theta_{yyx}\xi^2+\Theta_{yyy}\xi^3\big)\mathrm{d}z,\qquad \tilde\omega^4=\xi\mathrm{d}z+\mathrm{d}w,$$
$$\tilde\omega^5=\mathrm{d}y-\xi\mathrm{d}x-\big(\Theta_{xx}+2\xi\Theta_{xy}+\xi^2\Theta_{yy}\big)\mathrm{d}z.$$

Following [6], we now pass to the new coordinates $(\tilde x,\tilde y,\tilde z,\tilde p,\tilde t)$ on $\mathbb{T}(M^4)$

$$x\mapsto\tilde t,\qquad w\mapsto\tilde y\qquad z\mapsto\tilde x,\qquad -\xi\mapsto\tilde p,\qquad y\mapsto\tilde z-\tilde p\tilde t.$$

We obtain the distribution annihilated by the following 1-forms:

$$\tilde{\omega}^3 = -\mathrm{d}\tilde{p} - \tilde{A}\mathrm{d}\tilde{x}, \qquad \tilde{\omega}^4 = -\tilde{p}\mathrm{d}\tilde{x} + \mathrm{d}\tilde{y},$$
$$\tilde{\omega}^5 = \mathrm{d}\tilde{z} - \tilde{p}\mathrm{d}\tilde{t} - \tilde{t}\mathrm{d}\tilde{p} + \tilde{p}\mathrm{d}\tilde{t} - \tilde{B}\mathrm{d}\tilde{x} = \mathrm{d}\tilde{z} - \tilde{t}\mathrm{d}\tilde{p} - \tilde{B}\mathrm{d}\tilde{x} = \mathrm{d}\tilde{z} + (\tilde{t}\tilde{A} - \tilde{B})\mathrm{d}\tilde{x},$$

where $\tilde{A}$ and $\tilde{B}$ are coordinate transforms of the functions

$$A(w,x,y,z,\xi) = \Theta_{xxx} + 3\Theta_{yxx}\xi + 3\Theta_{yyx}\xi^2 + \Theta_{yyy}\xi^3,$$
$$B(w,x,y,z,\xi) = \Theta_{xx} + 2\xi\Theta_{xy} + \xi^2\Theta_{yy}$$

respectively. This suggests taking

$$\tilde{A} = -H'(t), \tilde{B} = -H(t)$$

to obtain the Legendre transformed 1-forms in Section 6.3. Passing back to coordinates (w, x, y, z, ξ) on $\mathbb{T}(M^4)$, this gives

$$-H'(x) = \Theta_{xxx} + 3\Theta_{yxx}\xi + 3\Theta_{yyx}\xi^2 + \Theta_{yyy}\xi^3,$$
$$-H(x) = \Theta_{xx} + 2\xi\Theta_{xy} + \xi^2\Theta_{yy},$$

so that

$$\Theta(x) = -\iint H(x)\mathrm{d}x\mathrm{d}x$$

will satisfy the condition. We have $\Theta_{xx} = -H(x)$. We have the following theorem.

Theorem 7.1. *The An–Nurowski twistor distribution $\mathcal{D}$ on the circle twistor bundle $\mathbb{T}(M^4) \to M^4$ of (M^4, g) with the Plebański metric*

$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y + H(x)\mathrm{d}z^2$$

and the function $H(x)$ has split G_2 as its group of local symmetries provided that $H(x)$ is one of the following up to fractional linear transformations in s:

1. *The function $H(x)$ is given by*

$$H(x) = \iint \frac{\dot{s}^2}{s^{\frac{4}{3}}(s-1)^{\frac{4}{3}}}\mathrm{d}x\mathrm{d}x,$$

where $s(x)$ is a solution to the 3^{rd} order ODE (5.2)

$$\{s, x\} + \frac{\dot{s}^2}{2}V(s) = 0$$

with (α, β, γ) given by either $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$ or $\left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

2. *The function $H(x)$ is given by*

$$H(x) = \iint \frac{\dot{s}^2}{s^{\frac{4}{3}}(s-1)}\mathrm{d}x\mathrm{d}x,$$

where $s(x)$ is a solution to (5.2) with (α, β, γ) one of $\left(\frac{2}{3}, \frac{1}{2}, \frac{1}{3}\right)$, $\left(\frac{2}{3}, \frac{1}{2}, \frac{4}{3}\right)$ or $\left(\frac{2}{3}, 2, \frac{1}{3}\right)$.

3. *The function* $H(x)$ *is given by*

$$H(x) = \iint \frac{\dot{s}^2}{s^{\frac{5}{3}}(s-1)^{\frac{5}{3}}} \mathrm{d}x\mathrm{d}x,$$

where $s(x)$ *is a solution to* (5.2) *with* (α,β,γ) *either* $\left(\frac{8}{3},\frac{2}{3},\frac{2}{3}\right)$ *or* $\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$.

4. *The function* $H(x)$ *is given by* $H(x) = x^m$ *where* $m \in \left\{-2,-\frac{1}{2},\frac{1}{2},2\right\}$.

5. *The function* $H(x)$ *is given by*

$$H(x) = -\frac{1}{192}\frac{\sqrt{x+C}(4x+3B+C)}{\sqrt{x+B}(B-C)^3}.$$

This corresponds to the solution obtained from (3.2).

6. *The function* $H(x)$ *is the Legendre transform of the function* $F(q)$ *with* $q = H'(x)$ *and*

$$H(x) = qx - F(q) = xH'(x) - F(H'(x)).$$

In this case $F(q)$ *can be given by one of the following:*

$$(a) \quad F(q) = \iint \frac{\dot{s}^{\frac{3}{2}}}{s(s-1)} \mathrm{d}q\mathrm{d}q$$

where $s(q)$ *is again a solution to the* 3^{rd} *order ODE* (5.2) *with* (α,β,γ) *one of* $(3,3,3)$ *or* $\left(3,\frac{1}{3},\frac{1}{3}\right)$.

$$(b) \quad F(q) = \iint \frac{\dot{s}^{\frac{3}{2}}}{s(s-1)^{\frac{3}{4}}} \mathrm{d}q\mathrm{d}q,$$

where $s(q)$ *is a solution to* (5.2) *with* (α,β,γ) *one of* $\left(\frac{3}{2},\frac{1}{3},\frac{1}{2}\right)$, $\left(\frac{3}{2},3,\frac{1}{2}\right)$ *or* $\left(\frac{3}{2},\frac{1}{3},\frac{9}{2}\right)$.

$$(c) \quad F(q) = \iint \frac{\dot{s}^{\frac{3}{2}}}{s^{\frac{5}{4}}(s-1)^{\frac{5}{4}}} \mathrm{d}q\mathrm{d}q,$$

where $s(q)$ *is a solution to* (5.2) *with* (α,β,γ) *one of* $\left(6,\frac{3}{2},\frac{3}{2}\right)$ *or* $\left(\frac{2}{3},\frac{3}{2},\frac{3}{2}\right)$.

$$(d) \quad F(q) = q^m,$$

where $m \in \left\{-1,\frac{1}{3},\frac{2}{3},2\right\}$.

$$(e) \quad F(q) = -\frac{1}{6}\frac{(q+B)^{\frac{1}{3}}(q+C)^{\frac{2}{3}}}{(B-C)^2},$$

again corresponding to the solution (3.2).

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